International Journal of Theoretical and Applied Mathematics
2022; 8(2): 40-44
http://www.sciencepublishinggroup.com/j/ijtam
doi: 10.11648/j.ijtam.20220802.12
ISSN: 2575-5072 (Print); ISSN: 2575-5080 (Online)

Generalized-function Solutions of a Differential Equation of L-order in the Space K'

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To cite this article:
Abdourahman. Generalized-function Solutions of a Differential Equation of L-order in the Space K'. *International Journal of Theoretical and Applied Mathematics*. Vol. 8, No. 2, 2022, pp. 40-44. doi: 10.11648/j.ijtam.20220802.12

Received: February 28, 2022; **Accepted**: March 24, 2022; **Published**: March 31, 2022

Abstract: The main purpose of this work is to study the existence of solutions (zero-centered solutions), in the sense of distributions, of the non-homogeneous l-order linear singular differential equation of the following type $\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \delta^{(s)}(x)$, where l is a natural number not equal to zero, s a natural number, $(a_i)_{0\leq i\leq l}$ real numbers and more $a_l\neq 0$, $k_i \in \mathbb{Z}_+$, $k_i \geq i$, $i = 0,1,\ldots\ldots,l.\ \delta^{(0)}(x) = \delta(x)$ is the Dirac distribution centered at 0 and $\delta^{(s)}(x)$ is the sth-order derivative of the Dirac delta function. For this aim, we apply some theorems and lemmas from the general concepts of theory of generalized functions in the work. Namely, we replace the general form of the particular solution $y(x) = \sum_{j=0}^{\gamma_0} C_j\ \delta(x)^{(j)}$ (as linear combination of Dirac delta functions and it derivatives) into the considered equation. This leads us to release the conditions of its solvency, formulated into a theorem and, let us analyze the algebraic system obtained for the determination of the unknown coefficients C_j. By this, we undertake the description of all zero-centered solutions of the considered equation into a theorem.

Keywords: Test Functions, Generalized Functions, Dirac Delta Function, Zero-Centered Solutions

1. Introduction

As we know a differential equation is an equation, where the unknown is a function and both the function and its derivatives may appear in the equation. We can stipulate that differential equations are essential for a mathematical description of nature as they lie at the core of many physical theories. We can illustrated that by just mentioning Newton's and Lagrange's equations for classical mechanics, Maxwell's equations for classical electromagnetism, Schr¨odinger's equation for quantum mechanics, and Einstein's equation for the general theory of gravitation.

It is well known that normal linear homogeneous systems of ODE with infinitely smooth coefficients have no generalized-function solutions other than the classical solutions see [13]. In contrast to this case, for equations with singularities in the coefficients, new solutions in generalized functions may appear and some classical solutions may disappear. The number γ_0 is called the order of the distribution.

$$y(x) = \sum_{j=0}^{\gamma_0} C_j\ \delta(x)^{(j)}.$$

As well known fact, the generalized-function solutions in the sense of distribution of ordinary differential equations can be derived from the theory of distributions. In these areas different investigations are conducted and still developping gradually and continually, consequently opening up many ways, aspects and properties in the theory of linear differential and functional differential equations.

It is also well known that the linear homogeneous ordinary differential equations with infinitely-smooth coefficients have no generalized solutions in the sense of the distribution; while the ordinary differential equations with polynomial coefficients such as the Cauchy-Euler equation defined by equation.

$\sum_{m=0}^{l} a_m(x)\frac{d^m y}{dx^m} = g(x)$, where $a_m(x) = c_m\, x^m$ with $c_m \in \mathbb{R}, c_l = 1\ and\ g(x) = 0$ may admit a classical solution or generalized solution in the sense of distribution and this is possible to find in [4-7] for more details.

We should notice that, many scientific papers recently are devoted to considerable interest in problems concerning the existence of solutions to differential and functional differential equations (FDE) in various spaces known of generalized functions.

Following S Jhanthanam, K Nonlaopon, and S Orankitjaroen in [11] and Abdourahman in [6], we can note that, the distributional solutions namely defined as series of Dirac delta function and its derivatives of some order have been used in various fields of applied mathematics such as the theory of partial differential equations, operational calculus, and functional analysis without forgoten quantum electrodynamics in the area of physics.

Here in this work, we investigate the question of the solvability of a linear singular differential equation of *l*-order with singularity and Dirac delta function (or it derivatives of some order) in the second right hand side in the space of generalized functions *K'*. Otherwise, we investigate the existence of solutions in the sense of distributions of such equation.

Namely, we consider the equation of the following kind:

$$\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \delta^{(s)}(x), \tag{1}$$

where l is a natural number not equal to zero, s a natural number, $(a_i)_{0\le i\le l}$ real numbers and more $a_l \neq 0$, $k_i \in \mathbb{Z}_+$, $k_i \geq i$, $i = 0,1,\ldots\ldots,l.\, \delta^{(0)}(x) = \delta(x)$ is the Dirac distribution centered at 0 and $\delta^{(s)}(x)$ is the *sth*-order derivative of the Dirac delta function.

Going in the same direction for the equation (1), we set in this paper only the problem of the research of all zero-centered solutions of this equation. We also underline that in the work, we focus our investigation only to the *Euler case* of the equation (1). We do not look for other kind of solutions in this work.

This paper is structured as follows: in section 2, we recall some fundamental well known concepts of distributions (generalized functions). Section 3 is discribing the *Euler case* and some results. Section 4 is devoted properly to the investigation of the solvability (existence of zero-centered solutions) of the considered equation in the situation called *Euler case*. We conclude our paper in section 5 after making an important remark before this conclusion.

2. Preliminaries

In this section we briefly review the notion of generalized functions (we refer to [1, 2, 3, 8, 10] and [11] for a detailed study).

Definition 2.1. Let K be the space consisting of all real-valued functions $\varphi(x)$ with continuous derivatives of all orders and compact support. The support of $\varphi(x)$ is the closure of the set of all elements $t \in \mathbb{R}$ $such\ that\ \varphi(x) \neq 0. Then, \varphi(x)\ is\ called\ a\ test\ function.$

Definition 2.2. A distribution T is a continuous linear functional on the space K on the space of the real-valued functions with infinitely-differentiable and bounded support. The space of all such distributions is denoted by K'.

For every $T \in K'$ and $\varphi(x) \in K$, the value T has on $\varphi(x)$ is denoted by $(T, \varphi(x))$. Note that $(T, \varphi(x)) \in \mathbb{R}$.

Below, let us give some examples of distributions.

A locally-integrable function g(x) is a distribution generated by the locally-integrable function g(x). Here, we define.

$$(g(x), \varphi(x)) = \int g(x)\varphi(x)dx\ integration\ on\ the\ support\ \Omega, and\ \varphi(x) \in K.$$

In this case the distribution is called regular distribution.

The Dirac delta function is a distribution defined by $(\delta(x), \varphi(x)) = \varphi(0)$, and the support of $\delta(x)$ is $\{0\}$.

In this case the distribution is called irregular distribution or singular distribution.

Definition 2.3. The sth-order derivative of a distribution T, denoted by $T^{(s)}$, is defined by $\left(T^{(s)}, \varphi(x)\right) = (-1)^s \left(T, \varphi(x)^{(s)}\right)$ for all $\varphi(x) \in K$.

Let give an exemple of derivatives of the singular distribution.

$T = \delta$ we have:

a) $(\delta'(x), \varphi(x)) = -(\delta(x), \varphi'(x)) = -\varphi'(0)$;

b) $\left(\delta^{(s)}, \varphi(x)\right) = (-1)^s \left(\delta(x), \varphi(x)^{(s)}\right) = (-1)^s\, \varphi(0)^{(s)}$.

Definition 2.4. Let $\omega(x)$ be an infinitely-differentiable function. We define the product of $\omega(x)$ with any distribution T in K' by $(\omega(x)T, \varphi(x)) = (T, \omega(x)\varphi(x))$ for all $\varphi(x) \in K$.

We conduct our research on the *l*-order linear singular differential equation of general form in the *Euler case*, focusing our investigations only on the existence of all zero-centered solutions for the nonhomogeneous equation (1).

Note that a simple equation $x^n y(x) = \delta^{(s)}(x)\ in\ the\ space\ K'$ admitting a distributional solution $\frac{\delta^{(s)}(x)}{x^n}$ is such that

$$y(x) \in K'\ and\ (x^n y(x), \varphi(x)) = \left(\delta^{(s)}(x), \varphi(x)\right), \varphi(x) \in K.$$

Therefore we find that the quotient (the division) of an *sth-order* derivative of the Dirac delta function $\delta^{(s)}(x)$ by an *nth-power* of $x\ i.e\ x^n$ is a distribution depending on *n* constants. It has been established by applying the Fourier Transform and its inverse to both sides of the equation, that this quotient is defined by the formula.

$$\frac{\delta^{(s)}(x)}{x^n} = \frac{(-1)^n s!}{(s+n)!}\delta^{(s+n)}(x) + \sum_{k=1}^{n} C_k \delta^{(k-1)}(x)$$

where C_k, $k = 1,\ldots\ldots n$ are arbitrary constants see [6]. This lead us to suppose that investigating some specific linear singular differential equations in the space of generalized functions *K'* is interesting and quite challenging as the solutions, in some particular cases, are expressed by rather huge and unexpected formulas.

Now, let move to the main result of our investigations conducted in the following section.

3. The Euler Case

In this part we investigate the case when the following equalities are satisfied.

$$k_i = k_0 + i;\ i = 0,1\ldots\ldots,l.$$

Further, in the next part for our investigations we will use the following definitions and lemmas.

Definition 3.1 The polynomial denoted.

$$P_l(j) = \frac{1}{j!}\sum_{i=0}^{l}(-1)^i a_i \ (j + k_0 + i)!$$

is called characteristic polynôm of the differential equation (1).

Definition 3.2 The set of zeros or roots of the defined polynomial $P_l(j)$ is denoted in the following way:

$$Nul\, P_l(j) = \{j_*: j_* \in \mathbb{Z}_+, P_l(j_*) = 0\} = KerP_l.$$

Let formulate the following needed lemma which can be found in some books related to generalized functions.

Lemma 3.1 Let $\omega(x)$ be an infinitely-differentiable function. Then

$$\omega(x)\delta(x)^{(s)} = (-1)^s\omega_{(0)}^{(s)}\delta(x) + (-1)^{s-1}s\omega_{(0)}^{(s-1)}\delta'(x) + +(-1)^{s-2}\tfrac{s(s-1)}{2!}\omega_{(0)}^{(s-2)}\delta''(x)+\ldots+\omega(0)\delta(x)^{(s)}, \quad (2)$$

and

$$[\omega(x)H(x)]^{(m)} = \omega_{(x)}^{(m)}H(x) + \omega_{(0)}^{(m-1)}\delta(x) + \omega_{(0)}^{(m-2)}\delta'(x)+\ldots\ldots+\omega(0)\delta(x)^{(m-1)}. \quad (3)$$

The proof of *Lemma 3.1* may be found in [9].

As useful formula that is deduced from (2), we can establish an important result for any monomial $\omega(x) = x^k$ in the following lemma.

Lemma 3.2. Let $k, s \in \mathbb{N} \cup \{0\}$. Then it is holding place.

$$x^k\delta(x)^{(s)} = \begin{cases} 0, s < k; \\ \dfrac{(-1)^k\, s!}{(s-k)!}\,\delta(x)^{(s-k)}, s \geq k. \end{cases}$$

The proof of this lemma can be found in some special mathematical books related to the theory of distributions, see also [8].

Now let move to the following interesting situation.

Below, we study the non-degenerated case and look for zero-centered solutions.

We consider the most interesting case when it is fulfilled the condition $\prod_{i=0}^{l} a_i \neq 0$ and call this case *non-degenerated case*.

The following theorem gives the necessary conditions of the existence of zero-centered solutions of the equation (1) in the space *K'*.

4. Main Results

In this section, we will state our main results and give their proofs.

Let formulate the following theorem.

Theorem 4.1: Let $\prod_{i=0}^{l} a_i \neq 0$; $k_i \in \mathbb{Z}_+$, $i = 1\ldots\ldots, l$; $k_0, s \in \mathbb{N}\cup\{0\}$.

For the existence of zero-centered solutions of the equation (1) in the space K', it is necessary and sufficient that:

$$\textstyle\sum_{i=1}^{l}(k_0 - k_i + i)^2 + (\sum_{i=0}^{l}(-1)^i a_i \ (k_0 + s + i)!)^2 \neq 0. \quad (4)$$

Proof. We just prove the necessary part of the theorem. The sufficient part will be deduced directly at the same while constructing the solutions.

It is clear that we should find the particular solution $y(x)$ of the nonhomogeneous equation (1) in the form of functional centered at zero, i.e:

$$y(x) = \textstyle\sum_{j=0}^{\gamma_0} C_j \ \delta(x)^{(j)}, \quad (5)$$

Where γ_0 is sufficient a large number. We suppose contrary, when

$$k_i = k_0 + i;\ i = 1\ldots\ldots, l;\ and\ \textstyle\sum_{i=0}^{l}(-1)^i a_i \ (k_0 + s + i)! = 0. \quad (6)$$

Simple calculations when taking into account *lemma 2.2* give us after setting (5) into the equation (1) the following result:

$$\textstyle\sum_{i=0}^{l} a_i\, x^{k_i} \sum_{j=0}^{\gamma_0} C_j \ \delta(x)^{(j+i)} = \sum_{i=0}^{l} a_i \ \sum_{j=0}^{\gamma_0} C_j \ x^{k_i}\delta(x)^{(j+i)} = \delta(x)^{(s)},$$

or that is the same as:

$$\textstyle\sum_{i=0}^{l} a_i \sum_{j=k_i-i}^{\gamma_0} C_j \ (-1)^{k_i} \frac{(j+i)!}{(j+i-k_i)!}\delta(x)^{(j+i-k_i)} = \delta(x)^{(s)}.$$

When changing $j + i - k_i$ into j we obtain

$$\textstyle\sum_{i=0}^{l} a_i \sum_{j=k_i-i}^{\gamma_0+i-k_i}(-1)^{k_i}C_{j+i-k_i} \frac{(j+k_i)!}{j!}\delta(x)^{(j)} = \delta(x)^{(s)} \quad (7).$$

Until now, the investigation has been conducted for the arbitrary k_i and the equality (7) is obtained without any supplementary suppositions. We next take into account $k_i = k_0 + i$ following the supposition (6) and we rewrite the system (7) in the form as:

$$\sum_{i=0}^{l} a_i \sum_{j=0}^{\gamma_0 - k_0} (-1)^{k_i} C_{j+k_0} \frac{(j+k_0+i)!}{j!} \delta(x)^{(j)} = \delta(x)^{(s)} \tag{8}$$

Or when changing the order of the summation we obtain:

$$\sum_{j=0}^{\gamma_0 - k_0} \delta(x)^{(j)} \left(\sum_{i=0}^{l} (-1)^{k_i} a_i \, (j + k_0 + i)! \right) \frac{C_{j+k_0}}{j!} = \delta(x)^{(s)} \tag{9}$$

From the previous by equalizing the coefficients under Dirac delta functions and it derivatives, we obtain a linear non homogeneous algebraic system for the determination of the unknown coefficients C_j of the following form:

$$(-1)^{k_0} \frac{C_{j+k_0}}{j!} \sum_{i=0}^{l} (-1)^i a_i \, (j + k_0 + i)! = \begin{cases} 1, j = s, \\ 0, j \neq s \end{cases} \tag{10}$$

It is not difficult to see that such equation when $j = s$ has the following form:

$$\frac{(-1)^{k_0}}{s!} C_{s+k_0} \sum_{i=0}^{l} (-1)^i a_i \, (s + k_0 + i)! = 1,$$

which contradicts the assumption made of the proved theorem. The theorem is proved.

Further, we consider the most simple and easy case when it is fulfilled the conditions:

$$k_i = k_0 + i; \; i = 0,1 \ldots\ldots, l, \sum_{i=0}^{l} (-1)^i a_i \;\; (s + k_0 + i)! \neq 0.$$

It takes place the following theorem:

Theorem 4.2 Let $\prod_{i=0}^{l} a_i \neq 0$; $k_i \in \mathbb{N}, i = 1 \ldots\ldots, l$; $k_0, s \in \mathbb{N} \cup \{0\}$ and be fulfilled the following conditions:

$$k_i = k_0 + i; \; i = 0,1 \ldots\ldots, l, \sum_{i=0}^{l} (-1)^i a_i \; (s + k_0 + i)! \neq 0, \tag{11}$$

then the general zero-centered solution of the equation (1) has the following form:

$$y(x) = \frac{(-1)^{k_0} s!}{\sum_{i=0}^{l} (-1)^i a_i \; (s+k_0+i)!} \delta(x)^{(s+k_0)} + \sum_{j=0}^{k_0 - 1} C_j \; \delta(x)^{(j)} \tag{12}$$

in the case, when

$$\sum_{i=0}^{l} (-1)^i a_i \;\; (j + k_0 + i)! \neq 0, \forall j \in \mathbb{Z}_+. \tag{13}$$

And if there exists at least one $j_*^m \in \mathbb{Z}_+ / \{s\}, m \leq l$, such that

$$j_*^m \in \; Nul \; P_l(j), \tag{14}$$

then the solution has the following form:

$$y(x) = \frac{(-1)^{k_0} s!}{\sum_{i=0}^{l} (-1)^i a_i \; (s+k_0+i)!} \delta(x)^{(s+k_0)} + \sum_{j=0}^{k_0 - 1} C_j \, \delta(x)^{(j)} + \sum_{j_*^m \in Nul \, P_l(j)} C_{j_*^m + k_0} \, \delta(x)^{(j_*^m + k_0)}. \tag{15}$$

Proof. As we notice upstair, in this case we can use the system (10) from which with the condition (11) we may find directly the following:

$$C_{s+k_0} = \frac{(-1)^{k_0} s!}{\sum_{i=0}^{l} (-1)^i a_i \; (s+k_0+i)!} \tag{16}$$

Concerning the other remaining coefficients C_{j+k_0}, j $\in \mathbb{Z}_+ / \{s\}$, so they are equal to zero, when $\sum_{i=0}^{l} (-1)^i a_i \;\; (j + k_0 + i)! \neq 0$ for all $j \in \mathbb{Z}_+$ and not more l from them will remain free, if there exist $j_*^m \in \mathbb{Z}_+ / \{s\}$, such that

$$\sum_{i=0}^{l} (-1)^i a_i \;\; (j_*^m + k_0 + i)! = 0.$$

All what has been said lead us to (12) - (15) with the consideration that $\delta(x)^{(j)}, j = 0,1 \ldots\ldots, k_0 - 1$ are solutions of the homogeneous equation.

The theorem is proved.

We want to remark immediatly if the conditions $k_i = k_0 + i; \; i = 0,1 \ldots\ldots, l$ are violated, then in the general situation the problem will be complicated, so that for the simplicity it is recommandable to start to investigate more simple case of the *first order* linear singular differential equation of the following form:

$$ax^p y'(x) + bx^q y(x) = \delta^{(s)}(x)$$

with $q > p - 1$ or $q < p - 1$ *in the same space* K'.

This will be one of the work that we can plan to investigate in a brief future to help us looking for generalizing the general similar cases when considering equation.

$$\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \delta^{(s)}(x)$$

with conditions $k_i = k_{i-1} + 1;\ i = 1\ldots\ldots, l-1$ but $k_l \neq k_{l-1} + 1$. We may call (in the two situations apearing) this equation left Euler case equation or right Euler case equation, depending of the cases $k_l > k_{l-1} + 1$ or $k_l < k_{l-1} + 1$.

Before concluding, let says that the function $y(x) \in K'$ is solution in the sense of distributions of the equation (1) if and only if:

$$\forall \varphi(x) \in K$$

$$\left(\sum_{i=0}^{n} a_i\, x^{k_i} y^{(i)}(x), \varphi(x)\right) = \left(\delta^{(s)}(x), \varphi(x)\right) = (-1)^s\, \varphi_{(0)}^{(s)}.$$

5. Conclusion

In this paper we completly investigated the question of the existence of zero-centered solutions and the solvability of an *l-order* linear singular differential equation in the *Euler case situation* in the space of generalized functions K' with a second right hand side in the form of Dirac delta function or it derivatives of some order. We applied well known theorems and lemmas for the investigation of solvability and the existence of zero-centered solutions of the consedered equation, when looking for the particular solutions in the form of linear combination of Dirac delta functions and it derivatives as $y(x) = \sum_{j=0}^{\gamma_0} C_j\, \delta(x)^{(j)}$. We analyzed the algebraic system appearing and gave the necessary and sufficient conditions of the existence of the solutions in theorem 4.1. Next, we undertook the determination of the unknown coefficients C_j and, therefore in theorem 4.2, we described all the obtained solutions depending of the relationships between the parameters of the equation. Finally, a remark is done concerning a situation when it is violated the *Euler equation case* conditions.

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