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Volume Integral Mean of Holomorphic Function on Polydisc

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Abstract: Let f be an analytic function in the Hardy space on the polydisc P_2. In this article we discuss the area integral means $M_p(f, r)$ of f on the polydisc P_2 with radius r, and its weighted volume means $M_{p,\alpha}(f, r)$ with to the weight $(1-|z_1|^2)^a\times(1-|z_2|^2)^a$. We prove that both $M_p(f, r)$ and $M_{p,\alpha}(f, r)$ are strictly increasing in r unless f is a constant. In contrast to the classical case, we also give a example to show that $\log M_{p,\alpha}(f, r)$ is not always convex with respect to $\log r$, although that we still prove that $\log M_p(f, r)$ is logarithmically convex.

Keywords: Hardy Space, Polydisc, Integral Means, Logarithmically Convex

1. The Introduction

Let f be an analytic function on unit circle D in complex plane $\mathbb{C}$. For $0<p<\infty$ the mean integral of f is defined as

$$M_p(f,r)=\left[\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta\right]^{1/p},0\le r<1, \qquad (1)$$

The classical Hardy convex theorem, which is an important tool for complex analysis and harmonic analysis, especially Hardy function space theory, states that $M_p(f,r)$ is strictly increasing in $r\in[0,1)$ and $\log M_p(f,r)$ is logarithmically convex (see [1]). Xiao and Zhu discussed the extension of these results in the volume integral case in [2]. In fact, they considered the more general unit sphere problem, namely,

$$M_{p,\alpha}(f,r)=\left[\frac{1}{v_\alpha(r\mathrm{B}_n)}\int_{r\mathrm{B}_n}|f(z)|^p dv_\alpha(z)\right]^{1/p},0\le r<1, \quad (2)$$

where α is real number and $dv_\alpha(z)=\left(1-|z|^2\right)^\alpha dv(z)$ is a weighted measure on the unit sphere, and concluded $M_{p,\alpha}(f,r)$ strictly increases in r and however $\log M_{p,\alpha}(f,r)$ of r is logarithmically convex or logarithmically concave depends on the sign of the parameter α. Further they proved that $\log M_{p,\alpha}(f,r)$ is logarithmically convex when $\alpha\le 0$, and a logarithmically concave function when is non-negative.

In this paper, we discuss the case on the polydisc $\mathrm{P}_2=\{|z_1|<1\}\times\{|z_2|<1\}$. Denote by $\partial_0\mathrm{P}_2=S_1\times S_2$ the characteristic boundary. Note that the topology of the characteristic boundary of the polydisc is very different from that of the unit sphere, therefore the function space on P_2 has some special properties (see [3]).

Let $f(z)$ be a holomorphic function on P_2 and $d\sigma(z)$ be a normalized Borel measures on a unit disk. For $0<p<\infty$, the mean of weighted volume integral of $f(z)$ is defined by

$$M_{p,\alpha,\beta}(f,r)=\left[\frac{1}{v_\alpha(r\mathrm{P}_n)}\int_{r\mathrm{P}_n}|f(z)|^p dv_{\alpha,\beta}(z)\right]^{1/p},0\le r<1, \quad (3)$$

where $dv_{\alpha,\beta}(z)=d\sigma_{\alpha(z)}(z_1)d\sigma_{\beta(z)}(z_2)$ with

$$d\sigma_{\alpha(z)}(z_1)=\left(1-|z_1|^2\right)^{\alpha} d\sigma(z_1),$$
$$d\sigma_{\beta(z)}(z_2)=\left(1-|z_2|^2\right)^{\alpha} d\sigma(z_2), \tag{4}$$

and

$$v_{\alpha,\beta}(r\mathrm{P}_2)=\int_{r\mathrm{P}_2} dv_{\alpha,\beta}(z)$$

We will only discuss the case when $\alpha=\beta$ in this article. We shall use the notation v_{α} instead of $v_{\alpha,\alpha}$ and $M_{p,\alpha}(f,r)$ instead of $M_{p,\alpha,\ \beta}(f,r)$. In particular, when $p=\infty$, it can be understood as

$$M_{\infty}(f,r)=\sup\{|f(z)|: z\in \mathrm{P}_2, |z|=r\} L^p(\mathrm{P}_2, dv_{\alpha}), 0\le r<1$$

in the usual sense. When $0<p<\infty$ and $\alpha>1$, the weighted Bergman space A_{α}^{p} is the intersection of holomorphic function spaces $\mathrm{H}(\mathrm{P}_2)$ and $L^p(\mathrm{P}_2, dv_{\alpha})$ on P_2, where the norm on A_{α}^{p} is given by

$$\|f\|_{p,\alpha}=\left[\int_{\mathrm{P}_2}|f(z)|^p dv_{\alpha}(z)\right]^{1/p}$$

We need some knowledge of the Hardy space. For $0<p<\infty$, the unit polydisc P_2 is a Hardy space, and H^p is a holomorphic function that contains P_2 satisfies the following conditions:

$$\|f\|_p=\sup_{0\le r<1} M_p(f,r)<\infty,$$

Where

$$M_p(f,r)=\left[\int_{s_1\times s_2}|f(r\zeta)|^p d\sigma(\zeta)\right]^{1/p}$$

is the area integral mean of f and $d\sigma(\zeta)$ is the normalization Lebesgue measure of multi-cylinder surface $\partial\mathrm{P}_2$, namely $d\sigma(\zeta)=\frac{1}{(2\pi)^2}d\theta_1 d\theta_2$. It should pointed out here that the Hardy space on polydisc is quite different from those on spheres. For example, the modulus of functions of Hardy space on n-dimensional complex spheres is integral on real 2n−1 dimensional spheres, while the modulus of function of Hardy space on n-dimensional polydisc is integral on real n-dimensional torus. In this paper, we will discuss the monotonicity of $M_p(f,r)$ and $M_{p,\alpha}(f,r)$ and the logarithmical convexity of the corresponding logarithmic functions $\log M_p(f,r)$ and $\log M_{p,\ \alpha}(f,r)$. Since the symmetry of multiple cylinders is not as good as that of unit spheres, the method presented in [2] do not apply to here. But we find a new method to discuss it, and there will be some new conclusions.

2. Monotonicity of $M_p(f,r)$ and $M_{p,\alpha}(f,r)$ and Its Application

We will firstly consider the case of the Hardy space.

Theorem 2.1 Let $0<p<\infty$ and let $f(z)$ be a holomorphic function with extraordinary values on P_2. Then the function $r\mapsto M_p(f,r)$ is strictly increasing in the interval $[0,1)$.

Remark. The same theorem on the unit disc has been proved in [4]. This problem on multiple cylinders is not easy to be transformed into the unit disc. It is also di erent from the case on the unit sphere in [2]. The symmetry of multiple cylinders is insu cient to use slice. We need some special handling to prove it.

Proof: Suppose that $f(z)=f(z_1,z_2)$ is not a constant function. Then either $f_{z_1}(z_2)=f(z_1,z_2)$ is non-constant with respect to $z_2\in s_1$ for some fixed z_1, or $f_{z_2}(z_1)=f(z_1,z_2)$ is a non-constant with respect to $z_1\in s_1$ for some fixed z_2. Without loss of generality, we might consider the former case. Instead of dealing directly with $M_p(f,r)$, let's start with binary real functions of functions [5].

$$\begin{aligned} M_p^p(f,r_1,r_2)&=\int_{\partial\mathrm{P}_2}|f(r_1\zeta_1,r_2\zeta_2)|^p ds(\zeta) \\ &=\int_{s_1\times s_2}|f(r_1\zeta_1,r_2\zeta_2)|^p ds(\zeta_1)ds(\zeta_2) \\ &=\int_{s_2}\left(\int_{s_1}|f(r_1\zeta_1,r_2\zeta_2)|^p ds(\zeta_1)\right)ds(\zeta_2)\end{aligned} \tag{5}$$

where $0\le r_1,r_2<1$. For fixed r_2 and ζ_2, the inner integral

$$\int_{s_1}|f(r_1\zeta_1,r_2\zeta_2)|^p ds(\zeta_1)$$

in (5) is $f_{z_2}(z_1)=f(z_1,z_2)$, which corresponding to a unitary holomorphic function $M_p^p(f_{r_2\zeta_2},r_1)$. Hence $M_p^p(f,r_1,r_2)$ is increasing with respect to r_1, i. e.

$$\frac{\partial M_p^p(f,r_1,r_2)}{\partial r_1}\ge 0. \tag{6}$$

Similarly, $M_p^p(f,r_1,r_2)$ is an increasing function of r_2. In particular, since $f_{z_2}(z_1)=f(z_1,z_2)$ is a non-constant function, $M_p^p(f,r_1,r_2)$ is a strictly increasing function of r_2, so

$$\frac{\partial M_p^p(f,r_1,r_2)}{\partial r_2}>0. \quad (7)$$

Now from $M_p^p(f,r)=M_p^p(f,r_1,r_2)$ and (6),(7) we get

$$\frac{dM_p^p(f,r)}{dr}=\left(\frac{\partial M_p^p(f,r_1,r_2)}{\partial r_1}+\frac{\partial M_p^p(f,r_1,r_2)}{\partial r_2}\right)\Bigg|_{r_1=r_2=r}>0, \quad (8)$$

which implies that $M_p(f,r)$ is strictly increasing.

We continue to prove the following conclusion of $M_{p,\alpha}(f,r)$.

Theorem 2.2 For $0<p<\infty$ and α being a real number.

Let $f(z)$ be a holomorphic function on P_2. Then the function $r\mapsto M_{p,\alpha,\beta}(f,r)$ is an increasing function on the interval $[0,1)$. In particular, if f is non-constant, then f is strictly increasing.

Proof: We may rewrite the integral as follows

$$\begin{aligned}&\int_{r\mathrm{P}_2}|f(z)|^p dv_\alpha(z)\\&=\int_0^r\int_0^r\left(1-|\rho_1|^2\right)^\alpha\left(1-|\rho_2|^2\right)^\alpha\rho_1\rho_2\int_{s_1\times s_2}\left|f\left(\rho_1 e^{i\theta},\rho_2 e^{i\phi}\right)\right|^p\frac{d\theta d\phi}{\pi^2}d\rho_1 d\rho_2\end{aligned} \quad (9)$$

Then

$$\begin{aligned}\frac{d}{dr}\mathrm{I}(v)&=r(1-r^2)\int_0^r\left(1-|\rho_1|^2\right)^\alpha\rho_1\int_{\mathrm{T}^2}|f(\rho_1\zeta,r\zeta)|^p d\sigma(\zeta)d\rho_1\\&+r(1-r^2)\int_0^r\left(1-|\rho_2|^2\right)^\alpha\rho_2\int_{\mathrm{T}^2}|f(r\zeta,\rho_2\zeta)|^p d\sigma(\zeta)d\rho_2\end{aligned} \quad (10)$$

where $\frac{d\theta d\phi}{\pi^2}=d\sigma(\zeta')$ is on torus and $\zeta=(\zeta_1,\zeta_2)\in\mathrm{T}^2$.

Similarly, since

$$v_\alpha(r\mathrm{P}_2)=\int_{r\mathrm{P}_2}dv_\alpha(z)=4\int_0^r\int_0^r\left(1-\rho_1^2\right)^\alpha\rho_1\left(1-\rho_2^2\right)^\alpha\rho_2 d\rho_1 d\rho_2 \quad (11)$$

we have

$$\frac{d}{dr}v_\alpha(r\mathrm{P}_2)=\frac{4r}{\alpha+1}\left(1-r^2\right)^\alpha\left[-\left(1-r^2\right)^{\alpha+1}+1\right] \quad (12)$$

Then the derivative of the function $M_{p,\alpha}^p(f,r)$ with respect to r is given by

$$\begin{aligned}&4\int_0^r\int_0^r\left(1-\rho_1^2\right)^\alpha\rho_1\left(1-\rho_2^2\right)^\alpha\rho_2\left[\begin{aligned}&r\left(1-r^2\right)^\alpha\int_0^r\left(1-\rho_1^2\right)^\alpha\rho_1\int_{\mathrm{T}^2}|f(\rho_1\zeta_1,r\zeta_2)|^p d\sigma(\zeta)d\rho_1\\&+r\left(1-r^2\right)^\alpha\int_0^r\left(1-\rho_2^2\right)^\alpha\rho_2\int_{\mathrm{T}^2}|f(r\zeta_1,\rho_2\zeta_2)|^p d\sigma(\zeta)d\rho_2\end{aligned}\right]\\&-\int_0^r\int_0^r\left(1-|\rho_1|^2\right)^\alpha\left(1-|\rho_2|^2\right)^\alpha\rho_1\rho_2\int_{\mathrm{T}^2}|f(\rho_1\zeta_1,\rho_2\zeta_2)|^p d\sigma(\zeta)d\rho_1 d\rho_2\\&\left[4\int_0^r\left(1-\rho_1^2\right)^\alpha\rho_1 d\rho_1 r\left(1-r^2\right)^\alpha+4\int_0^r\left(1-\rho_2^2\right)^\alpha\rho_2 d\rho_2 r\left(1-r^2\right)^\alpha\right]\end{aligned} \quad (13)$$

where

$$\frac{d}{dr}\left(\frac{1}{v_\alpha(r\mathrm{P}_2)}\int_{r\mathrm{P}_2}|f(z)|^p dv_\alpha(z)\right)\sim v_\alpha(r\mathrm{P}_2)\frac{d\mathrm{I}(v)}{dr}-\mathrm{I}(v)\frac{dv_\alpha(r\mathrm{P}_2)}{dr}.$$

Since

$$\int_{\mathrm{T}^2}|f(\rho_1\zeta_1,r\zeta_2)|^p d\sigma(\zeta)>\int_{\mathrm{T}^2}|f(\rho_1\zeta_1,\rho_2\zeta_2)|^p d\sigma(\zeta),$$
$$\int_{\mathrm{T}^2}|f(r\zeta_1,\rho_2\zeta_2)|^p d\sigma(\zeta)>\int_{\mathrm{T}^2}|f(\rho_1\zeta_1,\rho_2\zeta_2)|^p d\sigma(\zeta),$$

it follows that

$$\frac{d}{dr}\left(\frac{1}{v_\alpha(r\mathrm{P}_2)}\int_{r\mathrm{P}_2}|f(z)|^p dv_\alpha(z)\right)\geq 0, 0\leq r<1, \quad (14)$$

and the equality in (14) holds only if f is a constant function.

Therefore, $M_{p,\alpha}^p(f,r)$ or $M_{p,\alpha}(f,r)$ is an increasing function with respect to r, and if f is not a constant function, it is strictly increasing.

3. Logarithmical Convexity

We now consider the logarithmic function $\log M_p(f,r)$ on the Hardy space.

Theorem 3.1 Let $0<p<\infty$ and f be a holomorphic function in the Hardy space $\mathrm{H}_p(\mathrm{P}_2)$. Then the function $\log M_p(f,r)$ is a convex function with respect to $\log r$ with $r\in(0,1)$.

Proof: It is similar to the proof of Theorem 1.6 in Duren's book [4], w suppose for any real number λ that

$$\begin{aligned}m_p(f,\lambda,r_1,r_2)&=\frac{r_1^\lambda}{2\pi}\frac{r_2^\lambda}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\left|f_{r_2\zeta_2}(r_1\zeta_1,r_2\zeta_2)\right|^p d\theta_1 d\theta_2\\&=\frac{1}{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}|r_1\zeta_1|^\lambda|r_2\zeta_2|^\lambda|f_z(r_1\zeta_1,r_2\zeta_2)|^p d\theta_1 d\theta_2.\end{aligned} \quad (15)$$

Then $M_p(f,r_1,r_2)=m_p(f,0,r_1,r_2)$ $|z_1|^{\lambda}|z_2|^{\lambda}|f_{z_2}(z_1,z_2)|^p$ is a pluriharmonic function on $\{(z_1,z_2)\in \mathrm{P}_2, r'<|z_1|,|z_2|<r''\}$, so it exists a pluriharmonic function $U(z_1,z_2)$ on $\{(z_1,z_2)\in \mathrm{P}_2, r'<|z_1|,|z_2|<r''\}$ and equal to $|z_1|^{\lambda}|z_2|^{\lambda}|f_{z_2}(z_1,z_2)|^p$ on the boundary. By the relationship between the subharmonic function and the harmonic function, we have

$$m_p(f,\lambda,r_1,r_2)\le \frac{1}{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}U\left(r_1e^{i\theta_1},r_2e^{i\theta_2}\right)d\theta_1 d\theta_2, r'<r_1,r_2<r'' \quad (16)$$

The second-order mixed partial derivative of the right-hand side of (16) is obtained and then the order of integral and partial derivative is exchanged to get

$$\frac{\partial}{\partial r_2}\frac{\partial}{\partial r_1}\left(\int_0^{2\pi}\int_0^{2\pi}U\left(r_1e^{i\theta_1},r_2e^{i\theta_2}\right)d\theta_1 d\theta_2\right)=\frac{\partial}{\partial r_2}\int_0^{2\pi}\frac{1}{r_1}\int_0^{2\pi}\frac{\partial}{\partial n_1}U\left(\zeta_1,r_2e^{i\theta_2}\right)ds_1 d\theta_2 \quad (17)$$

In the upper formula, $\partial/\partial n_1$ represents the normal partial derivative of the first variable, $\mathrm{U}_1\cos\theta_1 d\theta_1+\mathrm{U}_2\sin\theta_1 d\theta_1=\frac{1}{r_1}\frac{\partial \mathrm{U}}{\partial n_1}d(\theta_1 r_1)$ and ds. By green's formula and since U is harmonic, we have that the inner interal of (17) ia an univariate function of $r_2e^{i\theta_2}$. Let

$$\int_{|\zeta|=r_1}\frac{\partial U}{\partial n_1}ds=g\left(r_2e^{i\theta_2}\right),$$

then

$$\frac{1}{r_1}\frac{\partial}{\partial r_2}\int_0^{2\pi}g\left(r_2e^{i\theta_2}\right)d\theta_2=\frac{c}{r_1r_2}.$$

Hence

$$\frac{\partial}{\partial r_2}\frac{\partial}{\partial r_1}\left(\int_0^{2\pi}\int_0^{2\pi}U\left(r_1e^{i\theta_1},r_1e^{i\theta_2}\right)d\theta_1 d\theta_2\right)=\frac{c}{r_1r_2} \quad (18)$$

where $c>0$ is a constant. The right hand side of (18), (16) is the following function,

$$\mathrm{A}\log r_1\log r_2+\mathrm{B}_1\log r_1+\mathrm{B}_2\log r_2+c \quad (19)$$

among them, $\mathrm{A}>0$. Now by (18) and (19) we have

$$\frac{1}{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}U\left(re^{i\theta_1},re^{i\theta_2}\right)d\theta_1 d\theta_2=\mathrm{A}(\log r)^2+(\mathrm{B}_1+\mathrm{B}_2)\log r+c=\mathrm{A}(\log r)^2+\mathrm{B}\log r+c, r'<r<r''. \quad (20)$$

The above formula is obviously a convex function of $\log r$, so (16) yields that $m_p(f,\lambda,r,r)$ is logarithmically convex, that is,

$$\log m_p\left(f,\lambda,\frac{r'+r''}{2},\frac{r'+r''}{2}\right)<\frac{1}{2}\left[\log m_p(f,\lambda,r',r')+\log m_p(f,\lambda,r'',r'')\right] \quad (21)$$

Similar to the discussion in Xiao-Zhu [2], (21) and Duren [4, p. 10] implies that $\log m_p(f,r)$ is also a convex function of $\log r$, which completes the proof.

Remark. Inspired by the logaithmical convexity in Theorem 3.1 and the weightless integral mean $M_p(f,r)$, Xiao and Zhu naturally consider the logarithmic convexity of the volume integral mean $M_{p,\alpha}(f,r)$ of $\log r$ on a unit sphere (see [2]). They found that the logarithmical convexity problem is more complicated at this time, in fact, for some α, $M_{p,\alpha}(f,r)$ is logarithmically convex, while for others it is logarithmically concave. On multiple cylinders, the problem becomes much more complicated because of the correlation between the two parameters α and β. We now give some examples below.

Example 3.2. Let f be a holomorphic function on P_2. Then for $1\le p<\infty$, $\log M_{p,0}(f,r)$ is a convex function of $\log r$.

Proof: Notice first that we have

$$M_{p,0}(f,r)=\left[\int_{\mathrm{P}_2}|f(r\zeta)|^p dv(\zeta)\right]^{1/p},$$

and we may rewrite it as

$$M_{p,0}(f,r)=\int_0^1\rho_1 d\rho_1\int_0^1\rho_2 d\rho_2\int_{\partial \mathrm{P}_2}|f(r\rho_1\zeta_1,r\rho_2\zeta_2)|^p d\sigma(\zeta) \quad (22)$$

By Theorem 3.1, the logarithm of the innermost integral of (22) is logarithmically convex, so by the method in [6] we obtain that $\log M_{p,0}(f,r)$ is a convex function of $\log r$.

Example 3.3. For $p=2$, $\alpha=1$. Let $f(z)=z_1z_2$. Then $\log M_{p,\alpha}(f,r)$ is a concave function of $\log r$.

Proof: By [2, Example 10], the logarithms of volume mean integrals $M_{p,\alpha}(z_1,r)$ and $M_{p,\alpha}(z_2,r)$ on a unit disk are concave function of $\log r$. Then simple calculations give

$$\log M_{p,\alpha}(f,r) = \log M_{p,\alpha}(z_1,r) + \log M_{p,\alpha}(z_2,r).$$

Hence $\log M_{p,\alpha}(f,r)$ is concave function of $\log r$.

4. Conclusion

The proof of the main theorem (Theorem 3.1) is different from that of Xiao-Zhu [2, 7-15]. And we only work on the case of $\alpha = \beta$. The problem becomes much more complicated for the case of different because parameters α and β. In addition to the lack of symmetry on multiple cylinders mentioned in the proof which makes the tool of slice function unusable, it can also be seen from (20) that the proof here does not deal with the case on three-dimensional multiple cylinders.

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