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The Solvability of a New Boundary Value Problem with Derivatives on the Boundary Conditions for Forward-Backward Linear Systems Mixed of Keldysh Type in Multivariate Dimension

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Abstract: The solvability of the boundary value problem for linear systems of the mixed hyperbolic-elliptic equations of Keldysh type in the multivariate domain with the changing time direction are studied. Applying methods of functional analysis, "ε-regularizing", continuation by the parameter and by means of prior estimates, the existence and uniqueness of generalized and regular solutions of a boundary value problem are established in a weighted Sobolev space.

Keywords: Changing Time Direction, Weighted Sobolev Space, System Equations of Mixed Type, Weak, Strong and Regular Solution, Forward-Backward Linear Systems Mixed of Keldysh Type

1. Introduction

Interest of investigations of non-classical equations arises in applications in the field of hydro-gas dynamics, modeling of physical processes (see, e.g., [6], [7], [11], [12], [13], [18], [20],[21] and the references given therein).

Non-classical model is defined as the model of mathematical physics, which is represented in the form of the equation or systems of partial differential equations that does not fit into one of the classical types-elliptic, parabolic, or hyperbolic. In particular, non-classical models are described by equations of mixed type (for example, the Tricomi equation), degenerate equations (for example, the Keldysh equation or the equations of Sobolev type (e.g., the Barenblatt-Zsolt-Kachina equation), the equation of the mixed type with the changing time direction and forward-backward equations.

In recent years the attention of many scholars has turned to the study of well-posed boundary value problems for non-classical equations of mathematical physics, in particular, for forward-backward equations of the parabolic type (e.g., [16], [19] and the references given therein).

In the theory of boundary value problems for degenerate equations and equations of mixed-type, it is a well-known fact that the well-posedness and the class of its correctness essentially depend on the coefficient of the first order derivative (younger member) of equations (e.g., [3], [4], [8],[9], [14],[18] and the references given therein).

In the paper [8] it was introduced the new called Fichera's function, in order to identify subsets of the boundary of the domain where the boundary value problem for such kind of equations is posed, where it is necessary or not to specify the boundary condition. A namely boundary conditions depend from sign of the Fichera's function $\Phi(x)$.

In the work [3] (see, Chapter 1, p. 191-197 and Chapter 3 p. 239-245) and papers [14],[22] new boundary conditions (so called type of problem "E "in which some part of the boundary shall be exempt from the boundary conditions) were studied.

In the paper [17], [18] various Dirichlet problems which can be formulated for equations of Keldysh type, one of the two main classes of linear elliptic–hyperbolic equations were investigated. Open boundary conditions (in which data are prescribed on only part of the boundary) and closed boundary conditions (in which data are prescribed on the entire boundary) were both considered. Emphasis is on the formulation of boundary conditions for which solutions can be shown to exist in an appropriate function space.

Boundary value problems for equations of mixed hyperbolic-elliptic type with changing time direction had been studied details in [21]-[22].Great difficulties come into being in the investigation of linear systems of degenerate elliptic and hyperbolic equations.

In mathematical modeling, partial differential equations of the mixed type are used together with boundary conditions specifying the solution on the boundary of the domain. In some cases, classical boundary conditions cannot describe process or phenomenon precisely. Therefore, mathematical models of various physical, chemical, biological or environmental processes often involve non-classical conditions. Such conditions usually are identified as nonlocal boundary conditions and reflect situations when the data on the domain boundary cannot be measured directly, or when the data on the boundary depend on the data inside the domain. In this case, boundary condition in particularly, maybe given for some part of the boundary with derivatives. Consequently, in this paper considered boundary conditions corresponds to the so- called well-posed boundary condition of Fichera's and Keldysh an application new approaches form presentation. In numerical methods for solving these equations, the problem of stability has received a great deal of importance and attention.

Finally, the problem for the system of equations of mixed hyperbolic-elliptic of Keldysh type, including property of changing time direction has not been extensively investigated. Therefore in present paper we will study this problem.

2. Problem Statement, Notation and Preliminaries

Let G be a bounded domain in the Euclidean space R^n of the point $x=(x_1,...,x_n)$, including a part of hyper-plane $x_n=0$ and with smooth boundary $\partial G\in C^2$, $G^+=G\bigcap\{x_n>0\}$, $G^-=G\bigcap\{x_n<0\}$. The boundary of G^+ consists of a part of hyper-plane $x_n=0$ for $x_n>0$ and smooth surface ∂G^+. Analogically, the boundary G^- consists of a part of hyper-plane $x_n=0$ for $x_n<0$ and smooth surface ∂G^- .Assume that $D=G\times(-T,T),\ T>0;\ S=\partial G\times(-T,T),$ where $\Gamma=\partial D$ is a boundary of domain D. In the domain D consider the system of equations:

$$\left.\begin{aligned}&L_1(u,\upsilon)=k_1^{(1)}(t)u_{tt}+k_2(x)\Delta_x u+\sum_{i=1}^n a_{i1}^{(1)}(x,t)u_{x_i}+\sum_{i=1}^n a_{i2}^{(1)}(x,t)\upsilon_{x_i}\\&+b_{11}(x,t)u_t+b_{12}(x,t)\upsilon_t+c_{11}(x,t)u+c_{12}(x,t)\upsilon=f_1(x,t)\\&L_2(u,\upsilon)=k_1^{(2)}(t)\upsilon_{tt}-\Delta_x\upsilon+\sum_{i=1}^n a_{i1}^{(2)}(x,t)u_{x_i}+\sum_{i=1}^n a_{i2}^{(2)}(x,t)\upsilon_{x_i}\\&+b_{21}(x,t)u_t+b_{22}(x,t)\upsilon_t+c_{21}(x,t)u+c_{22}(x,t)\upsilon=f_2(x,t)\end{aligned}\right\},\qquad(2.1)$$

where the Δ_x is Laplace operator $\Delta_x=\frac{\partial^2}{\partial x_1^2}+...+\frac{\partial^2}{\partial x_n^2}$

Everywhere we will assume that the coefficients of the system of equation (2.1) are sufficiently smooth. Moreover, the conditions

$$tk_1^{(i)}(t)>0 \text{ for } t\neq 0,\ t\in(-T,T),\ i=1,2;\ x_nk_2(x)<0 \text{ for } x_n\neq 0,\ x=(x_1,...,x_n)\in G\in R^n$$

are satisfied. As far as is known that quadratic form of the equations of system (2.1) changes, then this system contains partitions degenerating elliptic, degenerating hyperbolic, mixed and composite type differential equations at the same time including changing direction time of variable in the domain D.

Assume the notations:

$$\Gamma_{-T}^+=\{(x,t)\in\Gamma: x_n>0,\ t=-T\},$$
$$\Gamma_{-T}^-=\{(x,t)\in\Gamma: x_n<0,\ t=-T\},$$
$$\Gamma_T^+=\{(x,t)\in\Gamma: x_n>0,\ t=T\},$$
$$\Gamma_T^-=\{(x,t)\in\Gamma: x_n<0,\ t=T\}\ S^+=\partial G^+\times[-T,T],$$
$$S^-=\partial G^-\times[-T,T],\ D^+=D\bigcap\{x_n>0\},\ D^-=D\bigcap\{x_n<0\}.$$

The boundary value problem Find the solution of system equations (1.1) in the domain $\bar{D}$, satisfying the conditions:

$$u|_\Gamma=0,\ u_t|_{\overline{\Gamma_T^+}}=0,\ u_t|_{\overline{\Gamma_{-T}^-}}=0,\qquad(2.2)$$

$$\upsilon|_\Gamma=0,\ \upsilon_t|_{\overline{\Gamma_T^+}}=0,\ \upsilon_t|_{\overline{\Gamma_{-T}^-}}=0.\qquad(2.3)$$

Remark 2.1. In this situation, the $\Gamma_{-T}^+,\Gamma_{-T}^-,\Gamma_T^+,\Gamma_T^-$ set are carriers as boundary conditions which depending on the signs $b_{22}(x,t)$, $b_{11}(x,t)$, $k_1^{(i)}(t), i=1,2,$ when the $2b_{22}-\mathrm{k}_{1t}^{(2)}(t)\le-\delta<0$, $2b_{11}-k_{1t}^{(1)}(t)\le-\delta<0$ conditions must be satisfied everywhere in D. Thus indicated boundary value problems for the system of equations (2.1) are putting in the form (2.2), (2.3) and setting the boundary conditions (2.2), (2.3) corresponds to and consistent with the approach cited above(e.g., [3], [4], [8],[17], [22],etc.).

By the symbol C_L we denote a class of twice continuously differentiable functions in the closed domain D, satisfying the boundary conditions (2.2) and (2.3), by $H_{1,L}(D)$, $H_{2,L}(D)$ in Sobolev's space with weighted spaces obtained by the class C_L which is closed by the norms:

$$\|u\|^2_{H_{1,L}(D)} = \int_D (u_t^2 + |k_2(x)| \sum_{i=1}^n u_{x_i}^2 + u^2) dD,$$

$$\|u\|^2_{H_{2,L}(D)} = \int_D (u_{tt}^2 + k_2^2(x) \sum_{i=1}^n u_{x_i x_i}^2 + |k_2(x)| \sum_{i=1}^n u_{x_i t}^2 + |k_2(x)| \sum_{i=1}^n u_{x_i}^2 + u_t^2 + u^2) dD,$$

respectively. Introduce, the space $W_2^k(D)$ Sobolev's with norm (e.g. [2],[5]):

$$\|u\|^2_{W_2^k(D)} = \|u\|^2_{K,D} = \int_D \sum_{|\alpha| \le k} |D^\alpha u|^2 \, dxdt,$$

$$|\alpha| = \alpha_0 + ... + \alpha_n, \; D^\alpha = D_0^{\alpha_0} ... D_n^{\alpha_n}, \; D_0 = \frac{\partial}{\partial t}, \; D_i = \frac{\partial}{\partial x_i}.$$

Since $k_2(x) \ne 0$ for $x_n \ne 0$, by the Sobolev's embedding theorems [2],[5] the functions from the spaces $H_{2,L}(D)$ will satisfy the boundary conditions (2.2), (2.3).

Lemma 2.1. Assume that the following conditions

(a) $2b_{11}(x,t) - k_{1t}^{(1)}(t) \le -\delta < 0$ for $t = 0$, $x \in G$;

(b) $2b_{22}(x,t) - k_{1t}^{(2)}(t) \le -\delta < 0$ for $t = 0$, $x \in G$;

(c) $\left|a_{i1}^{(1)}(x,t)\right|^2 \le M_1 |k_2(x)|$, $\left|a_{i1}^{(2)}(x,t)\right|^2 \le M_2 |k_2(x)|$

$\sum_{i=1}^n \left(a_{i1}^{(1)} - k_{2x_i}\right)^2 \le M_3 |k_2(x)|$, where M_1, M_2, M_3, M- are sufficiently large constants,

(d) $2c_{22}(x,t) - \sum_{i=1}^n a_{i2}^{(2)} - b_{22}(x,t) \le 0$, $(x,t) \in D$,

(e) $\left|a_{i2}^{(1)}\right|^2 \le M |k_2(x)|$,

(f) $x_n c_{11}(x,t) \ge 0$ for $x_n \ne 0$, $(c_{11}(x,t))'_t = c_{11t}(x,t) \ge 0$, $(x,t) \in D$

or $\alpha c_{22t} - \alpha_t c_{22} \ge 0$ $\alpha c_{11t} - c_{11}\alpha_t \ge 0$, $(x,t) \in D$ are holds. If $\alpha(x_n, t) = -tx_n - M_4$, where constant M_4 is sufficiently large, then for all functions $u(x,t)$, $\upsilon(x,t) \in C_L$ the following inequality

$$\int_{D^+} L_1(u,\upsilon)\alpha u_t dD^+ + \int_{D^+} \alpha \upsilon_t L_2(u,\upsilon) dD^+ + \int_{D^-} \alpha u_t L_1(u,\upsilon) dD^- + \int_{D^-} \upsilon_t L_2(u,\upsilon) dD^- \ge m\left(\|u\|^2_{H_{1,L}(D)} + \|\upsilon\|^2_{H_{1,L}(D)}\right), \quad (2.4)$$

holds true. Where the constant m is not dependent from functions $u(x,t)$ and $\upsilon(x,t)$.

Proof. Let $u(x,t)$, $\upsilon(x,t) \in C_L$ and consider the following integrals:

$$J_1 = \int_{D^+} L_1(u,\upsilon)\alpha u_t dD^+ + \int_{D^-} \alpha u_t L_1(u,\upsilon) dD^- \;;$$

$$J_2 = \int_{D^+} L_2(u,\upsilon)\alpha \upsilon_t dD^+ + \int_{D^-} L_2(u,\upsilon)\alpha \upsilon_t dD^- .$$

After integration by parts and allowing for boundary conditions of (2.2), (2.3) and taking into account nonnegative boundary integrals we get:

$$J_1 = \int_{D^+} L_1(u,\upsilon)\alpha u_t dD^+ + \int_{D^-} \alpha u_t L_1(u,\upsilon) dD^- \ge \ge \frac{1}{2} \int_{D^+} \left\{ \left[(2b_{11} - k_{1t}^{(1)}(t))\alpha + \alpha_t k_1^{(1)}(t)) u_t^2 + 2\sum_{i=1}^n (a_{i1}^{(1)}\alpha - (k_2\alpha)_{x_i}) u_{xi} u_t + k_2 \alpha_t u_{x_i}^2 \right] + \right.$$

$$\left. 2\sum_{i=1}^n a_{i2}^{(1)}(x,t) \upsilon_{x_i} \alpha u_t + b_{12}\alpha \upsilon_t u_{tt} + (\alpha c_{11t} - c_{11}\alpha_t) u^2 + c_{12}\alpha \upsilon u_t \right]\Big\} dD^+ - \int_{G^+} k_1^{(1)}(-T)\alpha u_t^2(x,-T) dx + + \int_{G-} k_1^{(1)}(T)\alpha u_t^2(x,T) dx$$

$$-\int_{S^+} k_2(x)\alpha u_x^2(x,t) dx + \int_{S^-} k_2(x)\alpha u_x^2(x,t) dx,$$

$$J_2 = \int_{D^+} \alpha \upsilon_t L_2(u,\upsilon) dD^+ + \int_{D^-} \alpha \upsilon_t L_2(u,\upsilon) dD^- \ge$$

$$\ge \frac{1}{2} \int_{D^{-+}} \left\{ \left[(2b_{22} - k_{1t}^{(2)}(t))\alpha - \alpha_t k_1^{(2)}(t)) \upsilon_t^2 - 2\sum_{i=1}^n (\alpha_{x_i} \upsilon_t \upsilon_{x_i} + \upsilon_{x_i}^2 \alpha_t) + (\alpha c_{22} - \alpha_t c_{22t}) \upsilon^2 \right.\right.$$

$$\left.\left. +2\sum_{i=1}^n (a_{i1}^{(2)} u_{x_i} \alpha \upsilon_t + a_{i2}^{(2)} \upsilon_{x_i} \alpha \upsilon_t) + 2c_{21} u \upsilon_t + 2b_{21} u_t \upsilon_t \right]\right\} dD^- - \int_{G^+} k_1^{(2)}(-T)\alpha u_t^2(x,-T) dx + \int_{G-} k_1^{(1)}(T)\alpha u_t^2(x,T) dx .$$

Now, using inequalities of Cauchy-Bunyakovskiy, Poincare and conditions of Lemma2.1 for coefficients of system equations (2.1), and taking into account the fact that, the coefficients $k_1^{(i)}(t), i = 1,2$ are homogeneous on the boundaries, and then , summarizing estimates for J_1 and J_2 obtains the validity of inequality (2.4).

Definition2.1. We say that $u(x,t)$ and $\upsilon(x,t)$ are regular solution of problem ((2.1)-(2.3)), if the functions $u(x,t), \upsilon(x,t) \in H_{2,L}(D)$ satisfy equation of (2.1) almost everywhere in domain D .

We need to seek new structure step of proof or non-classical method for solvability of problem ((2.1)-(2.3)). For this reason first of all, begin to formulate the theory of existence, first take the decaying system equations in the following form:

$$L_1(u) = k_1^{(1)}(t)u_{tt} + k_2(x)\Delta u + \sum_{i=1}^{n} a_{i1}^{(1)}u_{x_i} + b_{11}u_t + c_{11}u = f_1(x,t) . \tag{2.5}$$

$$L_2(\upsilon) = k_1^{(2)}(t)\upsilon_{tt} - \Delta\upsilon + \sum_{i=1}^{n} a_{i2}^{(2)}\upsilon_{x_i} + b_{22}\upsilon_t + c_{22}\upsilon = f_2(x,t) . \tag{2.6}$$

For proving solvability of the problem ((2.5), (2.2)) we use the method of "ε – regularization" and it is the fact that the hyper-plane $x_n = 0$ is a characteristic for equation (2.5). Therefore, we can consider the boundary value problem ((2.5), (2.2)) in the following form:

Boundary value problem 1. Find the solution of equation (2.5) in the domain D^+, satisfying the boundary conditions

$$u\big|_{\overline{\Gamma_{-T}^{+}}} = 0,\ u\big|_{\overline{\Gamma_{T}^{+}}} = 0,\ u_t\big|_{\overline{\Gamma_{T}^{+}}} = 0,\ u\big|_{s^+} = 0. \tag{2.7}$$

Boundary value problem 2. Find the solution of equation (5) in the domain D^-, satisfying the boundary conditions

$$u\big|_{\overline{\Gamma_{-T}^{-}}} = 0,\ u_t\big|_{\overline{\Gamma_{-T}^{-}}} = 0,\ u\big|_{\overline{\Gamma_{T}^{-}}} = 0\ \ u\big|_{s^-} = 0. \tag{2.8}$$

By $C_L'(D^+)$, $C_L'(D^-)$ we denote a class of infinitely differentiable functions in the closed domains D^+, D^- satisfying the boundary conditions (2.7) and (2.8), respectively.

3. Uniqueness Solution of Problem ((2.1)-(2.3)) in Space $H_{2,L}(D)$

Theorem3.1. Assume that the conditions of Lemma2.1 hold, then the regular solution of the problem ((2.1)-(2.3)) is unique.

Proof. Indeed, let u_1, υ_1 and u_2, υ_2 be two solutions of problem ((1)-(3)) which is satisfying the systems equations (1). Let $u = u_1 - u_2$, $\upsilon = \upsilon_1 - \upsilon_2$. Then the functions u, υ will be satisfying equations: $L_1(u,\upsilon) = 0$ and $L_2(u,\upsilon) = 0$ in the domain D. Suppose that, $u \neq 0, \upsilon \neq 0$ be satisfied. Let's take sequence, functions $\{u_n\},\{\upsilon_n\} \in C_L$, $n = 1,2,...$ etc, such that $u_n \to u$ in $H_{2,L}(D)$ for $n \to \infty$, $\upsilon_n \to \upsilon$ in $H_{2,L}(D)$ for $n \to \infty$. By the inequality of (2.4) we have

$$\left\|L_1(u,\upsilon)\right\|_{L_2(D)} + \left\|L_2(u,\upsilon)\right\|_{L_2(D)} \geq m_1\left(\|u\|_{H_{2,L}(D)} + \|\upsilon\|_{H_{2,L}(D)}\right)$$

where the constant m_1 independent from the functions $u(x,t)$ and $\upsilon(x,t)$. Therefore we can assert that $L_1(u_n,\upsilon_n) \to L_1(u,\upsilon)$, $L_2(u_n,\upsilon_n) \to L_2(u,\upsilon)$ for $n \to \infty$. By the virtue of inequality of (2.4) we have

$$\begin{aligned}&\left(L_1(u_n,\upsilon_n),\alpha u_{nt}\right)_{L_2(D^+)} + \left(L_1(u_n,\upsilon_n),\alpha u_{nt}\right)_{L_2(D^-)} + \left(L_2(u_n,\upsilon_n),\alpha\upsilon_{nt}\right)_{L_2(D^+)} + \\ &+\left(L_2(u_n,\upsilon_n),\alpha\upsilon_{nt}\right)_{L_2(D^-)} \geq m_1\left(\|u_n\|^2_{H_{1,L}(D)} + \|\upsilon_n\|^2_{H_{1,L}(D)}\right).\end{aligned}$$

Hence, passing to limit as $n \to \infty$ in last inequality, we get $u_n \to 0$, $\upsilon_n \to 0$ in space $H_{1,L}(D)$. On the other sides we have

$$\|u_n - u\|_{H_{1,L}(D)} \leq \|u_n - u\|_{H_{2,L}(D)} \to 0\ ,$$
$$\|\upsilon_n - \upsilon\|_{H_{1,L}(D)} \leq \|\upsilon_n - \upsilon\|_{H_{2,L}(D)} \to 0$$

for $n \to \infty$. Hence, $u \equiv 0, \upsilon \equiv 0$. That is proof of Theorem 3.1. Now, we need the proof of solvability problem ((2.1)-(2.3)).

4. The Existence Weak (Regular) Solution of Problems ((2.5), (2.7)) ((2.5), (2.8))

Lemma4.1. Assume that the condition (a)-(c), (e), (f) of Lemma2.1 are holds, then for any functions $u(x,t) \in C_L'(D^+)$ $\left(u(x,t) \in C_L'(D^-)\right)$ following inequalities

$$\left(L_1(u),\alpha u_t\right)_{L_2(D^+)} \geq m_1\|u\|^2_{H_{1,L}(D^+)}\ ,\left(L_1(u),\alpha u_t\right)_{L_2(D^-)} \geq m_2\|u\|^2_{H_{1,L}(D^-)} \tag{4.1}$$

are valid.

Proof. Let's consider the integrals:

$$\int_{D^+} L_1(u)\alpha u_t dD^+ = \int_{D^+} f_1\alpha u_t dD^+\ ,$$

$$\int_{D^-} L_1(u)\alpha u_t dD^- = \int_{D^-} f_1\alpha u_t dD^-.$$

After integration by parts, allowing for boundary conditions and taking into account nonnegative boundary integrals we get

$$\left(L_1\left(u_1\right),\alpha u_t\right)_{L_2(D^+)} \geq \frac{1}{2}\int_{D^+}\left\{\left[\left(2b_{11}(x,t)-k_{1t}^2\right)\alpha-k_1^{(1)}\alpha_t\right]u_t^2+\sum_{i=1}^n 2\left[\left(\alpha a_{i1}^{(1)}(x,t)-(k_2(x)\alpha)_{x_i}\right)u_{x_i}u_t+\right.\right.$$
$$\left.\left.+k_2\alpha_t u_{x_i}^2\right]+\left[(-c_{11}\alpha_t+c_{11t}\alpha\right]u^2\right\}dD-\int_{G^+}k_1^{(1)}(-T)\alpha u_t^2(x,-T)dx \quad \forall u(x,t)\in C_L'(D^+),$$

$$\left(L_1\left(u_1\right),\alpha u_t\right)_{L_2(D^-)} \geq \frac{1}{2}\int_{D^-}\left\{\left[\left(2b_{11}(x,t)-k_{1t}^2\right)\alpha-k_1^{(1)}\alpha_t\right]u_t^2+\sum_{i=1}^n\left[2\left(\alpha a_{i1}^{(1)}(x,t)-(k_2(x)\alpha)_{x_i}\right)u_{x_i}u_t+\right.\right.$$
$$\left.\left.+k_2\alpha_t u_{x_i}^2\right]+\left[(-c_{11}\alpha_t+c_{11t}\alpha\right]u^2\right\}dD+\int_{G^-}k_1^{(1)}(T)\alpha u_t^2(x,T)dx \;,\; \forall u(x,t)\in c_L'(D^-)\,.\,\forall u(x,t)\in c_L'(D^-)\,.$$

Hence, using Cauchy-Bunyakovskiy and Poincare inequalities, taking into account conditions (a) – (c), (e),(f) of Lemma 2.1, for chosen constants with the fact that coefficients $k_1^{(1)}(t)$, is homogeneous on the boundaries ,then we get the truth of inequalities (4.1). Moreover, using inequality Holder's we have

$$\|f_1\|_{L_2(D^+)} \geq m_1\|u\|_{H_{1,L}(D^+)}^2 \;,\; \|f_1\|_{L_2(D^-)} \geq m_2\|u\|_{H_{1,L}(D^-)}^2$$

where the constants m_1, m_2 are independent from the function $u(x,t)$ .That is proof of Lemma 4.1

Definition4.1.The function $u(x,t)\in H_{2,L}(D^+)$ $\left(u(x,t)\in H_{2,L}(D^-)\right)$ is said to be regular solution of problem ((2.5), (2.7)), ((2.5), (2.8)) if it is generalized solution satisfies almost everywhere equation (2.5) in domain D^+ (D^-).

Lemma 4.2. Let the conditions of Lemma 4.1 be fulfilled. Then regular solution of problem ((2.5), (2.7)), ((2.5), (2.8)) is unique.

Proof. The Lemma4.2 is proved similarly way to the Lemma2.1 and Lemma 4.1.Since the equation of (2.5) is also degenerating then, due to regularizing effect to apply for equation (2.5)

In the domain D^+ , "$\varepsilon-$regularized" equation of mixed type

$$L_{1\varepsilon}\left(u_\varepsilon\right)=k_1^{(1)}(t)u_{\varepsilon tt}+(k_2-\varepsilon)\Delta u_\varepsilon+b_{11}u_{\varepsilon t}+\sum_{i=1}^n a_{i1}^{(1)}u_{\varepsilon x_i}+c_{11}u_\varepsilon=f_1\left(x,t\right) \tag{4.2}$$

and we state for it the boundary value problem

$$u_\varepsilon\big|_{x_n=0}=0\,,\; u_\varepsilon\big|_{S^+}=0\,,\; u_\varepsilon\big|_{\overline{\Gamma_T^+}}=0\,,\; u_\varepsilon\big|_{\overline{\Gamma_{-T}^+}}=0\;\; u_{\varepsilon t}\big|_{\overline{\Gamma_T^+}}=0. \tag{4.3}$$

Analogically, we will consider the following boundary value problem

$$L_{1\varepsilon}\left(u_\varepsilon\right)=k_1^{(1)}(t)u_{\varepsilon tt}+(k_2+\varepsilon)\Delta u_\varepsilon+b_{11}u_{\varepsilon t}+\sum_{i=1}^n a_{i1}^{(1)}u_{\varepsilon x_i}+c_{11}u_\varepsilon=f_1\left(x,t\right) \tag{4.4}$$

$$u_\varepsilon\big|_{x_n=0}=0\,,\; u_\varepsilon\big|_{S^-}=0\,,\; u_\varepsilon\big|_{\overline{\Gamma_{-T}^-}}=0\,,\; u_\varepsilon\big|_{\overline{\Gamma_T^-}}=0\;\; u_{t\varepsilon}\big|_{\overline{\Gamma_{-T}^-}}=0. \tag{4.5}$$

Proceeding from the known results of the papers [4], we can affirm the following proposition.

Remark4.2. If the conditions of Lemma 4.1, Lemma 4.2 and $2b_{11}(x,t)-\left|k_{1t}^{(1)}\right|\leq-\delta<0\;(x,t)\in D$ are satisfied, then for any right-hand side $f_1(x,t)$, $f_{1t}(x,t)\in L_2(D^+)$ $\left(f_1(x,t), f_{1t}(x,t)\in L_2(D^-)\right)$ there exists a unique solution of boundary value problem (4.2), (4.3) ((4.4), (4.5)) from the space $W_2^2(D^+)$ $\left(W_2^2(D^-)\right)$ and this solution allows following estimates

$$\|f_1\|_{L_2(D^+)}^2+\|f_{1t}\|_{L_2(D^+)}^2 \geq m_3\|u_\varepsilon\|_{W_2^2(D^+)}^2 \;,\|f_1\|_{L_2(D^-)}^2+\|f_{1t}\|_{L_2(D^-)}^2 \geq m_4\|u_\varepsilon\|_{W_2^2(D^-)}^2 \tag{4.6}$$

where the constants m_3 and m_4 are independent of the function $u(x,t)$.

Proof of this proposition proves similarly to Lemma2.1, Lemma4.1 and Theorem3.1.

Theorem 4.1. (*on the solvability of problem (* (2.5), (2.7)) Assume that the conditions of Lemma4.1 hold. If $\left|k_{2x_i}k_{2x_j}\right|\leq M_1\left|k_2(x)\right|, f_1(x,t), f_{1t}(x,t)\in L_2(D^+)$, $2b_{11}(x,t)-\left|k_{1t}^{(1)}(t)\right|\leq-\delta<0,$ for $(x,t)\in D^+$, $i,j=1,2,...,n$ are satisfied, then there exists a unique regular solution of problem ((2.5), (2.7)) from the space $H_{2,L}(D^+)$.

Theorem 4.2. (*on the solvability of problem* (2.5), (2.8)) Assume that the conditions of Lemma2.1 hold. If $\left|k_{2x_i}k_{2x_j}\right|\leq M_1\left|k_2(x)\right|, f_1(x,t), f_{1t}(x,t)\in L_2(D^-)$,

$2b_{11}(x,t)-\left|k_{1t}^{(1)}(t)\right|\le-\delta<0,\ (x,t)\in D^-$, $i,j=1,2,...,n$ are satisfied, then there exists a unique regular solution of problem ((2.5), (2.8)) from the space $H_{2,L}(D^-)$.

Proof of Theorem4.1 and 4.2 The following a priori estimates

$$\begin{aligned}\|f_1\|_{L_2(D^+)} &\ge m_5\int_{D^+}\left(u_{\varepsilon t}^2+|k_2-\varepsilon|\sum_{i=1}^n u_{\varepsilon x_i}^2+u_\varepsilon^2\right)dD^+\\ \|f_1\|_{L_2(D^-)} &\ge m_6\int_{D^-}\left(u_{\varepsilon t}^2+|k_2+\varepsilon|\sum_{i=1}^n u_{\varepsilon x_i}^2+u_\varepsilon^2\right)dD^-\end{aligned} \tag{4.7}$$

hold for the functions $u_\varepsilon(x,t)\in W_2^2(D^+)$ $\left(u_\varepsilon(x,t)\in W_2^2(D^-)\right)$ and being the solution of boundary value problems ((4.2), (4.3)), ((4.4), (4.5)), respectively. Where the constants m_5 and m_6 are independent of ε and $u(x,t)$. The proof of these statements in easily obtained by integration by parts and using the Cauchy inequality. Further for obtaining the second a priori estimation we take the function $\xi_1(t)$ such that

$$\xi_1(t)\equiv1 \text{ for } t\in(-T,-\eta),\ \frac{T}{2}>\eta>0;$$

$$\xi_1(t)\le1 \text{ for } t\in\left[-\eta,-\frac{\eta}{2}\right)\ \xi_1(t)\equiv0 \text{ for } t\in\left[-\frac{\eta}{2},T\right).$$

Then, we consider the function $W_\varepsilon(x,t)=\xi_1(t)u_\varepsilon(x,t)$. Obviously, the function $W_\varepsilon(x,t)$ will satisfy the equation

$$L_{1\varepsilon}W_\varepsilon=\xi_1 f_1+2k_1^{(1)}(t)\xi_1'(t)u_{\varepsilon t}+k_1^{(1)}(t)\xi_1''(t)u_\varepsilon=F_\varepsilon\,. \tag{4.8}$$

Hence, by virtue of inequalities (4.6) and (4.7), the set of functions $\{F_\varepsilon(x,t)\}$ are uniformly bounded in space $L_2(D^+)$. In other side, in domain $D^+_{\frac{\eta}{2}}=\left\{x\in D,-T<t<-\frac{\eta}{2}\right\}$ the equation $L_{1\varepsilon}W_\varepsilon=F_\varepsilon$ belongs to elliptical type of equation. Therefore, multiply equation of (4.8) by $-W_{\varepsilon tt}$ integrate by parts in the domain D^+, allowing boundary conditions, use the Cauchy-Bunyakovskiy inequality we get

$$\|F_\varepsilon\|_{L_2(D^+)}\ge m_7\int_{D^+}\left(W_{\varepsilon tt}^2+W_\varepsilon^2+W_{\varepsilon t}^2+|k_2-\varepsilon|\sum_{i=1}^n W_{\varepsilon x_i}^2+|k_2-\varepsilon|\sum_{i=1}^n W_{\varepsilon x_i x_i}^2\right)dD^+,$$

where constant m_7 is independent of ε, $u(x,t)$. Now, let's consider the function $\xi_2(t)\in C^\infty(-T,T)$ such that $\xi_2(t)\equiv0$ for $-T<t<-2\eta$, $\xi_2(t)\equiv1$, $-\eta<t<T$. Since $0\le\xi_2(t)\le1$ and $\xi_1(t)+\xi_2(t)\equiv1$, then taking $\phi_\varepsilon(x,t)=\xi_2(t)u_\varepsilon(x,t)$. It is easy to see that the functions $\phi_\varepsilon(x,t)$ satisfy the equation

$$L_{1\varepsilon}\phi_\varepsilon=\xi_2(t)f_1(x,t)+2k_{1t}^{(1)}(t)\xi_2'(t)u_\varepsilon(x,t)+k_1^{(1)}(t)\xi_2''(t)u_\varepsilon=\Phi_\varepsilon(x,t)\ . \tag{4.9}$$

Hence include that the functions $\Phi_\varepsilon(x,t),\Phi_{\varepsilon t}(x,t)$ are uniformly bounded with respect to ε in the space $L_2(D^+)$. Therefore, we can take finite difference

$$\phi_{\varepsilon h}=\frac{\phi_\varepsilon(x,t+h)-\phi_\varepsilon(x,t)}{h}$$

It is easy to see that the function $\phi_\varepsilon(x,t)$ satisfy the equations

$$L_{1\varepsilon}(\phi_{\varepsilon h})=\xi_2(t)f_1+2k_1^{(1)}(t)\xi_2'(t)u_{\varepsilon t}+k_1^{(1)}(t)\xi_2''(t)u_\varepsilon=\Phi_{\varepsilon h}(x,t)$$

Using the results on smoothness of the solution of problem ((4.2), (4.3)) and a priori estimates (4.6), (4.7) and passing to limit as $h\to0$ in the obtained inequalities

$$\|\varphi_{\varepsilon h}\|_{L_2(D^+)}\ge m_8\int_{D^+}\left(\phi_{\varepsilon hh}^2+\phi_\varepsilon^2+\phi_{\varepsilon h}^2+|k_2-\varepsilon|\sum_{i=1}^n\phi_{\varepsilon x h_i}^2+|k_2-\varepsilon|\sum_{i=1}^n\phi_{\varepsilon x_i x_i}^2\right)dD^+$$

and establishing relation between the functions $f_1(x,t)$ and $\Phi_\varepsilon(x,t)$ we get

$$\|f_1\|_{L_2(D^+)}+\|f_{1t}\|_{L_2(D^+)}\ge m_9\left(\int_{D^+}\left(u_{\varepsilon tt}^2+|k_2-\varepsilon|\sum_{i=1}^n u_{\varepsilon t x_i}^2+u_{\varepsilon t}^2+|k_2-\varepsilon|\sum_{i=1}^n u_{\varepsilon x_i}^2+u_\varepsilon^2\right)dD^+,\forall u_\varepsilon(x,t)\in C_L'(D^+)\right).$$

From the representations of function $\phi_\varepsilon(x,t)$ and from equation (4.2) by standard estimation method, we get

$\left|k_2-\varepsilon\right|\sum_{i=1}^{n}u_{\varepsilon x_i x_i}\in L_2\left(D^+\right)$. Hence, by standard compactness method we can conclude that $u\left(x,t\right)$ is generalized solution of problem ((2.5), (2.7)) and belongs to the space $H_{2,L}\left(D^+\right)$ and at the same time satisfy the equation (2.5) and condition (2.7) almost everywhere. In a similar way, repeating all the steps carried out for the domain D^+ for D^- also we can establish that problem ((2.5), (2.8)) has a generalized solution and belongs to the space $H_{2,L}\left(D^-\right)$.

5. Main Result of Existence and Uniqueness Strong (Regular) Solution of Problems ((2.5), (2.7)) ((2.5), (2.8))

Definition5.1. (following [1],[4],[10]) The function $u\left(x,t\right)\in H_{1,L}\left(D^+\right)$ $\left(u\left(x,t\right)\in H_{1,L}\left(D^-\right)\right)$ is said to be a strong solution of boundary value problem (10), (11) ((12), (13)), if there exists a sequences of functions $\left\{u_{\varepsilon n}\right\}\in C_L'\left(D^+\right)$ $\left(\left\{u_{\varepsilon n}\right\}\in C_L'\left(D^-\right)\right)$ such that equality

$$\lim_{n\to\infty}\left\|L_1\left(u_n\right)-f_1\left(x,t\right)\right\|_{L_2\left(D^+\right)}=\lim_{n\to\infty}\left\|u_n-u\right\|_{H_{1,L}\left(D^+\right)}=0\,,$$

is fulfilled in the domain D^- as well if instead of the domain taken D^+.

The following theorem on the existence of strong solution holds.

Theorem 5.1. Assume that the conditions of Lemma 2.1 hold. If

$$\left|k_{2x_i}k_{2x_j}\right|\le M_1\left|k_2\left(x\right)\right|,\quad i,j=1,...,n\,,$$
$$2b_{11}-\left|k_1^{(1)}\left(t\right)\right|\le-\delta<0,\quad \left(x,t\right)\in D$$

are satisfied , then for any function $f_1\in L_2\left(D^+\right)$ $\left(f_1\in L_2\left(D^-\right)\right)$ there exists a unique strong solution of boundary value problem (2.5), (2.7) from the space $H_{1,L}\left(D^+\right)$ (for the problem (2.5), (2.8) from $H_{1,L}\left(D^-\right)$).

Proof. From these Theorem 3.1, Theorem 4.1, Theorem4.2 there exists $u^+\left(x,t\right)$ solution of problem ((2.5), (2.7)), $u^-\left(x,t\right)$ solution of problem ((2.5), (2.8)) in the domains D^+ and D^-, respectively, and belonging respectively to the spaces $H_{2,L}\left(D^+\right)$ and $H_{2,L}\left(D^-\right)$. Then by the construction of such spaces there exists sequences $\left\{u_n\right\}\in C_L'\left(D^+\right)$ $\left(\left\{u_n\right\}\in C_L'\left(D^-\right)\right)$ such that

$$\lim_{n\to\infty}\left\|u_n^+-u^+\right\|_{H_{2,L}\left(D^+\right)}=\lim_{n\to\infty}\left\|u_n^--u^-\right\|_{H_{2,L}\left(D^-\right)}=0\,.$$

From the obvious inequality

$$\left\|u_n^+\right\|_{H_{2,L}\left(D^+\right)}\ge m\left\|L_1\left(u_n^+\right)\right\|_{L_2\left(D^+\right)},\ \left\|u_n^-\right\|_{H_{2,L}\left(D^-\right)}\ge m\left\|L_1\left(u_n^-\right)\right\|_{L_2\left(D^-\right)}$$

it follows that $\left\{L_1\left(u_n^+\right)\right\}\to f_1^+$ in $L_2(D^+)$, for $n\to\infty$. $\left\{L_1\left(u_n^-\right)\right\}\to f_1^-$ in $L_2(D^-)$, for $n\to\infty$.. Thus, suppose that $f_{1t}^+\in L_2\left(D^+\right)$, $f_{1t}^-\in L_2\left(D^-\right)$, then regular solutions u^+ and u^- are strong solution. We are constructing the sequences of functions $f_{1n}^+\in W_2^1\left(D^+\right)$, $f_{1n}^-\in W_2^1\left(D^-\right)$ such that $\left\{f_{1n}^+\right\}\to f_1^+$ in $L_2(D^+)$, $\left\{f_{1n}^-\right\}\to f_1^-$ in $L_2(D^-)$, for $n\to\infty$ .Then for the functions f_1^+ and f_1^- there exists strong solution problem of ((2.5), (2.7)) and ((2.5), (2.8))from the space $H_{2,L}\left(D^+\right)$ and $H_{2,L}\left(D^-\right)$ respectively. So, by inequality of Lemma 2.1 we have

$$\left\|f_{1n}^+\right\|_{L_2\left(D^+\right)}\ge m\left\|u_n^+\right\|_{H_{1,L}\left(D^+\right)},\ \left\|f_{1n}^-\right\|_{L_2\left(D^-\right)}\ge m\left\|u_n^-\right\|_{H_{1,L}\left(D^-\right)}.$$

Hence, we can include that $u_n^+\to u^+$ in $H_{1,L}\left(D^+\right)$, $u_n^-\to u^-$ in $H_{1,L}\left(D^-\right)$, for $n\to\infty$ and these functions are strong of problem ((5), (7)) and((5), (8)) respectively.

6. The Solvability of Problem ((2.5), (2.2))

Theorem 6.1. (Gluing solutions in the spaces) Assume that $u^+\in H_{i,L}\left(D^+\right)$, $u^-\in H_{i,L}\left(D^-\right)$,i=1,2,hold,then the constructed function

$$u\left(x,t\right)=\begin{cases}u^+\left(x,t\right),\ \left(x,t\right)\in D^+,\\ u^-\left(x,t\right),\ \left(x,t\right)\in D^-\end{cases}\tag{6.1}$$

will also be from the class $u\left(x,t\right)\in H_{i,L}\left(D\right),\ i=1,2$.

Proof. The Theorem6.1 proved exactly and similarly way to the Remark6.1 (e .g. [22]) **.**

Thus, we have the proof of the following theorem accordance essentially a combination of the proof of Theorems 3.1, 4.1, 4.2 and Lemmas 2.1, 4.1, 4.2 and Theorem 6.1.

Now, we can proof the main theorem of solvability of problem ((2.5), (2.2)).

Theorem6.2. (*On the solvability of problem* ((5), (2))) Assume that the conditions of Lemma2.1, Lemma4.1 and Theorems 3.1, 4.1, 4.2, 5.1 are satisfied, then for any functions $f_1, f_{1t}\in L_2\left(D\right)$ there exists a unique generalized solution of problem ((2.5), (2.2)) from the space $H_{2,L}\left(D\right)$.

Proof. Since on the base of Theorem 4.1, Theorem 4.2 and Theorem 5.1 there exists a unique solution $u^+\left(x,t\right)$, $u^-\left(x,t\right)$ of problems ((2.5), (2.7)) and ((2.5), (2.8))from the space $H_{2,L}\left(D^+\right)$ and $H_{2,L}\left(D^-\right)$ respectively. Then function $u\left(x,t\right)$ which is constructed by formula (6.1) will also be from the class $u(x,t)\in H_{2,L}(D)$ and at the same time is generalized solution of equation (2.5), moreover, the functions $u^{+}(x,t)$ and $u^{-}(x,t)$ is strong generalized solution of problem ((2.5), (2.2)).Consequently, it means that the strong and weak solutions of corresponding problems are identity (e.g. [1],[10]). It follows that the problem ((2.5), (2.2)) is solvability. The uniqueness of problem ((2.5), (2.2)) follows by means of inequality of Lemma 2.1. That is proof of Theorem 3.1. Analogically, the existence strong solution of problem ((2.5), (2.2)) from the space $H_{1,L}(D)$ can be proved.

7. On the Solvability of Problem ((2.1)-(2.3))

For proving the solvability of problem ((2.1)-(2.3)) we use the method of "continuation by parameter". It holds.

Theorem 7.1. (*on the solvability problem of (2.6), (2.3)*) Assume that the conditions

$$2c_{22}(x,t)-\sum_{i=1}^{n}a_{i2}^{(2)}(x,t)-b_{22}(x,t)\le-\delta_1<0,\quad (x,t)\in D,\quad (7.1)$$

$$\left|a_{i1}^{(2)}(x,t)\right|^2\le M_1\left|k_2(x)\right| \quad (7.2)$$

holds, then for any functions of $f_2(x,t),f_{2t}(x,t)\in L_2(D)$ there exists unique solution of problem ((2.6), (2.3)) in the space $H_{2,L}(D)$. (in case , instead of condition of (7.2), replaced smallest of coefficient $\left|a_{i1}^{(2)}(x,t)\right|$,then there exists unique solution of problem ((2.6), (2.3)) in space $W_2^2(D)$).

Proof. By virtue of condition (7.1) and $2b_{22}-\left|k_{1t}^{(2)}\right|\le-\delta<0$ the operator

$$L_2(\upsilon)=k_1^{(2)}(t)\upsilon_{tt}-\Delta\upsilon+\sum_{i=1}^{n}a_{i2}^{(2)}\upsilon_{x_i}+b_{22}\upsilon_t+c_{22}\upsilon$$

is coercive. Since the coefficient of $k_1^{(2)}(t)$ is sign fixed (according to [4]), then there exists unique solution of problem ((2.6), (2.3)) in space $W_2^1(D)$. If $\upsilon(x,t)\in W_2^1(D)$ then, (accordance to [15]) any solution of problem ((2.6), (2.3)) will be element of space $W_2^2(D)$. Analogically, repeating all the steps carried out for the solution $\upsilon(x,t)\in H_{2,L}(D)$ and also we can establish that problem ((2.6), (2.3)) has generalized solution if the condition (7.1) is satisfied. Therefore the theorem 7.1 is proved. Now we must prove solvability of problem ((2.1)-(2.3)). Let

$$M\bar{u}=K\bar{u}_{tt}+\sum_{i=1}^{n}A_i\bar{u}_{x_i}+B\bar{u}_t+C\bar{u}\,,\quad N\bar{u}=\sum_{i=1}^{n}P_i\bar{u}_{x_i}+Q\bar{u}_t+R\bar{u}$$

where $K=\begin{pmatrix}k_1^{(1)}&0\\0&k_1^{(2)}\end{pmatrix}$, $A_i=\begin{pmatrix}a_{i1}^{(1)}&0\\0&a_{i2}^{(2)}\end{pmatrix}$, $B=\begin{pmatrix}b_{11}&0\\0&b_{22}\end{pmatrix}$, $C=\begin{pmatrix}k_2\Delta+c_{11}&0\\0&\Delta+c_{22}\end{pmatrix}$, $P_i=\begin{pmatrix}0&a_{i2}^{(1)}\\a_{i1}^{(2)}&0\end{pmatrix}$, $Q=\begin{pmatrix}0&b_{12}\\b_{21}&0\end{pmatrix}$, $R=\begin{pmatrix}0&c_{12}\\c_{21}&0\end{pmatrix}$, $\bar{u}=\begin{pmatrix}u\\\upsilon\end{pmatrix}$, $\bar{f}=\begin{pmatrix}f_1\\f_2\end{pmatrix}$.

Then the system equations (1) can be written in the form:

$$L\bar{u}=M\bar{u}+N\bar{u}=\bar{f}\ . \quad (7.3)$$

Theorem7.2. Assume that the conditions of Lemmas 2.1, 4.1, 4.2 and Theorems 3.1, 4.1, 4.2, 5.1, 6.1, 6.2 7.1 moreover $f_1,f_2,f_{1t}f_{2t}\in L_2(D)$, $f_2(x,-T)=0$, $\left|a_{i2}^{(1)}\right|^2\le M\left|k_2(x)\right|$ are fulfilled, then there exists a unique solution of problem (2.1)-(2.3) in space $H_{2,L}(D)$. In case of $a_{i2}^{(1)}$ is smallest then there exists a unique solution of problem ((2.1)-(2.3)) from the space $H_{2,L}(D)\cap W_2^2(D)$.

Proof. Multiply the equation (7.3), by the vector $\overline{\eta_1}=(\alpha u_t,-\upsilon)$ in domain D, after integration by parts and using the Cauchy inequality, allowing for boundary condition (by analogically action to the Lemma2.1) we get the following estimates

$$\left\|L\bar{u}\right\|_{L_2(D)}\ge m\left\|\bar{u}\right\|_{H_{1,L}(D)}\ \text{or}\ \left\|L\bar{u}\right\|_{L_2(D)}\ge m\left\|\bar{u}\right\|_{H_{1,L}(D)\cap W_2^1(D)} \quad (7.4)$$

Now, let $H_{t,0}$ - is the space of vector function $\bar{\phi}=(\phi_1,\phi_2)$ such that $\phi_1,\phi_{1t},\phi_2\in L_2(D)$ and $\phi_1(x,-T)=0$ .The norm of space $H_{t,S}$ is defined by $\|\phi\|_{t,0}^2=\|\phi_{1t}\|_0^2+\|\phi_2\|_0^2$

From the results of the theorems 6.1, 7.1 it follows the following a prior estimates

$$\left\|\bar{u}\right\|_{H_{2,L}(D)}\le m_6\left\|M\bar{u}\right\|_{t,0}\ \text{or}\ \left\|\bar{u}\right\|_{H_{2,L}(D)\cap W_2^2(D)}\le m_7\left\|M\bar{u}\right\|_{t,0} \quad (7.5)$$

where m, m_6 , m_7 constant are not dependent from $\bar{u}(x,t)$ .It remains to show that, analogical estimates(7.4), (7.5) are also have to for operator $L\bar{u}$. Indeed, we may rewrite $M\bar{u}=L\bar{u}-N\bar{u}$, then

$$\left\|\bar{u}\right\|_{H_{2,L}(D)}\le m_8\left(\left\|L\bar{u}\right\|_{t,0}+\left\|N\bar{u}\right\|_{t,0}\right)\ \text{or}\ \left\|\bar{u}\right\|_{H_{2,L}(D)\cap W_2^2(D)}\le m_9\left(\left\|L\bar{u}\right\|_{t,0}+\left\|N\bar{u}\right\|_{t,0}\right)$$

are valid. Now, we consider the set of equations: $L_\tau\bar{u}=M\bar{u}+\tau N\bar{u}$ where $0\le\tau\le1$. Obviously, the following a prior estimate is uniformly bounded respect to parameter of τ : $\left\|\bar{u}\right\|_{H_{2,L}(D)}\le m_{10}\left\|L_\tau\bar{u}\right\|_{t,0}$ where m_{10} independent from parameter τ and $\bar{u}(x,t)$. Other side for $\tau=0$ we have $L_0\bar{u}=M\bar{u}$. In this case were considered problem is solvable. Notice that if $\tau=1$ then $L_1=L$ .Then as well as known method of continuation by parameter (for example, see [15],etc.) with the standard approaches the solvability of problem (2.1) ,(2.3) ,(2.4) can be proved.

Author suggests the open problem (7.6), (7.7):

$$Lu \equiv K_1(t)u_{tt} + K_2(x)u_{xx} + a(x.t)u_t + b(x,t)u_x + c(x,t)u + |u|^{\rho} u = f(x,t) \quad (7.6)$$

$$tK_1(t) > 0, t \neq 0, t \in (-1,1),\ xK_2(x) < 0, x \neq 0, x \in (\alpha,\beta), \alpha < 0, \beta > 0, \rho > -1,$$

the coefficients of equation (7.6) are sufficiently smooth .

The boundary value problem. Find the solution of equation (7.6) in domain $\bar{D} = \{\alpha \le x \le \beta, -1 \le t \le 1\}$, satisfying the conditions:

$$u\big|_{x=\alpha} = 0, u\big|_{x=\beta} = 0, u\big|_{t=-1,x>0} = 0, u\big|_{t=1,x<0} = 0 . \quad (7.7)$$

8. Conclusion

The solvability of the boundary value problem for linear systems of the mixed hyperbolic-elliptic type in the multivariate domain with the changing time direction are studied. The existence and uniqueness of generalized and regular solutions of a boundary value problem are established in a weighted Sobolev space. In this case applying idea of result works (e.g., [1], [10],[22]) , and Theorem 6.1,6.2, 7.1, 7.2 prove that weak and strong solutions of the boundary value problem for linear systems equations of the mixed hyperbolic-elliptic type in the multivariate domain with the changing time direction are identity. .

References

[1] L. Sarason, On weak and strong solutions of boundary value problems, Comm. PureAppl. Math. 15 (1962), 237-288. MR27 #460.

[2] Adams R. Sobolev Spaces, Second Ed., Academic Press, Elsevier Science, 2003.

[3] A.V. Bitsadze, Some Classes of Partial Differential Equations, Gordon and Breach: New York, 1988.

[4] V.N. Vragov, Boundary Value Problems for the Nonclassical Equations of Mathematical Physics, Novosibirsk: NSU, 1983. (in Russian).

[5] S. L. Sobolev, Applications of Functional Analysis in Mathematical Physics; English transl. Amer. Math. Soc, Providence, R.I., 1963.

[6] L. Bers, Mathematical Aspects of Subsonic and Transonic Gas Dynamics, Surveys in Applied Mathematics, vol. 3, John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958.

[7] F. I. Frankl, Selected Works in Gas Dynamics, Nauka, Moscow, Russia, 1973.

[8] G. Fichera,“On a unified theory of boundary value problems for elliptic-parabolic equations of second order, ”Boundary Problems in Differential Equations, Univ. of Wisconsin Press, Madison, pp. 97–120, 1960.

[9] Friedrichs, K. O. Symmetric positive linear differential equations. Comm. Pure Appl. Math. 11 (1958), 338–418.

[10] Friedrichs, K. O. The identity of weak and strong differential operators. Trans. Amer. Math. Soc. 55 (1944), 132–151.

[11] S. Canic, B. L. Keyfitz, E. H. Kim, “Mixed hyperbolic-elliptic system in self-similar flows,” Bol. Soc. Brasil. Mat.(N.S.) vol. 32, no. 3, pp. 377–399, 2001.

[12] C. Somigliana. “Sui sisteme simmetrici di equazioni a derivate parziali,” Ann. Math. Pure et Appl., II, v. 22, pp.143-156, 1894.

[13] B. Pini, Un Problem Di Valoru ol Contorno Por un’equazional a Derivative Puzziali Def Terro Ardine Con Parto Principale Di Tipo Composite, Rend. Sem. Fas. Sci. Univ. Gagliaro, 27, 114, 1957.

[14] M.V. Keldysh, “ On certain classes of elliptic equations with singularity on the boundary of the domain, ,” Dokl. Akad. Nauk SSSR, 77, pp. 181–183, 1951.

[15] O.A.Ladyzhenskaya, (English translation: The Boundary Value Problems of Mathematical Physic, Applied Mathematical Sciences, 49. Springer- Verlag, New York, 1985).

[16] Tersenov S.A. About a forward-backward equation of parabolic type. Novosibirsk, Nauka, 1985 (in Russian) .

[17] T.H. Otway, The Direchlet Problem For Elliptic-Hyperbolic Equations of Keldysh Type, Lecture Notes in Mathematics ISSN print edition:0075-8434, Springer Heidelberg Dordecht, London, New York, 2012.

[18] D. Lupo, C.S. Morawetz, K.K. Peyne, “On closed boundary value problems for equations of elliptic-hyperbolic type”, Commun. Pure. Appl. Math., vol. 60, pp.1319-1348, 2007.

[19] La’kin N.A, Novikov V.A.and Yonenko N.N. Nonlinear equations of variable type. Novosibirsk, 1983, Nauka.

[20] C. S. Morawetz,” A weak solution for a system of equations of elliptic-hyperbolic type,” Comm. Pure Appl. Math. Vol.11, pp. 315–331, 1958

[21] Nurmamedov M.A. On the solvability of the first local boundary value problems for linear systems equations of non-classical type with second order . Journal Doklad (Adigey) International Academy, Nalchik, 2008, v.10, №2, p.51-58 (in English)

[22] Nurmammadov M.A. The Existence and Uniqueness of a New Boundary Value Problem (Type of Problem “E”) for Linear System Equations of the Mixed Hyperbolic-Elliptic Type in the Multivariate Dimension with the Changing Time Direction. Hindavi Publishing Cooperation, Abstract and Applied Analysis Volume 2015, Research Article ID 7036552 pp. 1-10 (in English)