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Optimum Solutions of Fredholm and Volterra Integro-differential Equations

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Abstract: Integro-differential equations arise in modeling various physical and engineering problems. Several numerical and analytical methods have been developed for solving integro-differential equations. In this paper, a powerful semi analytical technique known as Optimal Homotopy Asymptotic Method (OHAM) has been used for finding the approximate solutions of Fredholm type integro-differential equations and Volterra type integro-differential equations. The proposed method does not required discretization like other numerical and approximate method, and it is also free from any small/large parameters. The presented technique provides better accuracy at lower order of approximation, the accuracy of the method can further be increases with higher order of approximation. Moreover, we can easily adjust and control the convergence region. The ability of the method is checked with different problems in literature. The results obtained through OHAM are compared with solutions of Adomian Decomposition Method. It is observed that solutions obtained through the proposed method is more accurate than existing techniques, which proves the validity and stability of the proposed method for solving integro-differential equations. The presented technique is more consistent, effective, suitable and rapidly convergent. The use of Optimal Homotopy Asymptotic Method is simple and straight forward. For the computation of problems, we have used Mathematica 9.0.

Keywords: Integro-differential Equations, Approximate Solutions, Optimal Homotopy Asymptotic Method

1. Introduction

Numerous mathematical formulations of physical phenomena involve integro-differential equations (IDE), these equations arise in many fields of science like chemical kinetics, fluid dynamics and biological models. In fact, obtaining the exact solutions of integro-differential equations are usually problematic so it is mandatory to attain efficient approximate solutions. Different approaches in literature have been used for the solutions of these equations. Some of them are Variational Iterative Method (VIM) [1-2], Modified Homotopy Perturbation Method (MHPM) [3-4], Homotopy Analysis Transform Method (HAM) [5-8], Conjugate Gradient Method (CGM) [9], Adomian Decomposition Method (ADM) [10-11] and Block Pulse Functions (BPF) [12] etc. Optimal Homotopy Asymptotic method (OHAM) is one of the most power full techniques for finding the approximate solutions of differential and integro differential equations. The method was introduced by Marinca et al. [13, 14] for the solution of differential equations. Many authors extended the proposed method for the solution of different kinds of integral equations. Khan et al. [15] used the proposed method for the solution of Volterra integral equation of the first kind. Almousa et al. [16] applied it for the solution of linear Fredholm integral equations of the first kind. Hashmi et al. [17] implemented the proposed method for the solution of Fredholm integro-differential equations.

In this article, we apply the proposed method for finding the approximate solutions of Volterra and Fredholm integro-differential equations. The general nth order integro-differential equations [18] has the following form

$$\psi^{n}(\mathrm{s})+\sum_{i=0}^{n-1}\psi^{i}(\mathrm{s})f_{i}(s)+\int_{a}^{b}k(s,t)\psi^{m}(t)\,dt=g(\mathrm{s}),\quad \mathrm{a}<\mathrm{s}<\mathrm{b},$$

with initial conditions

$$\psi(a)=\alpha_0,\ \psi'(a)=\alpha_1,\ \psi''(a)=\alpha_2,\ \ldots\ \psi^{(n-1)}(a)=\alpha_{n-1},$$

where $\alpha_i's$ are real constants, m and n are integers such that $m<n$, $f_i's$, g and k are given functions and ψ is the solution to be determined. The error analysis of the numerical problems confirms convergence and stability of the proposed method.

This paper is divided into five main sections. In section 2, basic idea of OHAM is introduced. Section 3 consists of numerical problems. Section 4 is the results and discussion, while in section 5 some conclusions are drawn.

2. Basic Idea of OHAM

In this section, the formulation of OHAM is presented, consider equation of the form:

$$L(\psi(s))+g(s)+N(\psi(s))=0,\quad B\left(\psi,\frac{d\psi}{ds}\right)=0. \qquad (1)$$

where L is for linear and N for non-linear operator, $\psi(s)$ is unknown function and $g(s)$ is known function while B is boundary operator. We obtain the family of equations by introducing an embedding parameter $p\in[0,1]$ as follows:

$$(1-p)\{L(\psi(s,p))+g(s)\}=H(p)\{L(\psi(s,p))+g(s)+N(\psi(s,p))\}, \qquad (2)$$

$$B\left(\psi(s,p),\frac{d\psi(s,p)}{ds}\right)=0.$$

where $\psi(s,p)$ is unknown function, and for $p\neq 0$ the non-zero auxiliary function $H(p)$ is given as

$$H(p)=c_1p^1+c_2p^2+c_3p^3+\ldots..+c_mp^m,$$

where $c_1,c_2,c_3,\ldots.,c_m$ are auxiliary constants and for $p=0$, $H(p)=0$. When $p=0$ or $p=1$ clearly, we have

$$p=0\Rightarrow H(\psi(s,0),0)=L(\psi(s,0))+g(s)=0, \qquad (3)$$

$$p=1\Rightarrow H(\psi(s,1),1)=H(1)\{L(\psi(s,p))+g(s)+N(\psi(s,p))\}=0. \qquad (4)$$

Obviously when $p=0$ and $p=1$. It keeps that $\psi(s,0)=\psi_0(s)$, and $\psi(s,1)=\psi(s)$, respectively. So as p varies from 0 to 1, the result $\psi(s,p)$ approaches from $\psi_0(s)$ to $\psi(s)$, where $\psi_0(s)$ is obtained Eq (2). for $p=0$:

$$L(\psi_0(s))+g(s)=0,\beta\left(\psi_0,\frac{d\psi_0}{ds}\right)=0. \qquad (5)$$

The Taylor's series expansion about p for obtaining the approximate solution, we write as follows:

$$\psi(s,p;c_i)=\psi_0(s)+\sum_{\kappa=1}^{\infty}\psi_\kappa(s,c_i)p^\kappa,\quad i=1,2,\ldots \qquad (6)$$

when $p=1$, one has written the series (6) as

$$\psi(s,1;c_i)=\psi_0(s)+\sum_{\kappa=1}^{\infty}\psi_\kappa(s,c_i),\quad i=1,2,\ldots \qquad (7)$$

when Eq. (6) is substituted in Eq. (2) and the coefficients of like power of p is compare, we obtain the equation of $\psi_0(s)$ given by (5) and the governing equations of $u_k(s)$ is obtained as follows:

$$L(\psi_1(s))=c_1N_0(\psi_0(s)),\quad B\left(\psi_1,\frac{d\psi_1}{ds}\right)=0,$$

$$L(\psi_k(s)-\psi_{k-1}(s))=c_kN_0(\psi_0(s))+\sum_{i=1}^{k-1}c_i\begin{bmatrix}L(\psi_{k-i}(s))\\+N_{k-i}(\psi_0(s),\psi_1(s),\ldots,\psi_{k-1}(s))\end{bmatrix},\ B\left(\psi_k,\frac{d\psi_k}{ds}\right)=0,\ k=2,3,\ldots, \qquad (8)$$

where $N_m(\psi_0(s),\psi_1(s),\ldots.,\psi_m(s))$ is the coefficient of p^m, obtained by expanding $N(\psi(s,p;c_i))$ in series with respect to the embedding parameter p; For $i=1,2,\ldots.,$

$$N(\psi(s,p;c_i))=N_0(\psi_0(s))+\sum_{k=1}^{\infty}N_k(\psi_0,\psi_1,\ldots,\psi_k)p^k, \qquad (9)$$

where $\psi(s,p;c_i)$ is given by Eq. (6) It should be noted that ψ_k for $k\geq 0$ is given by the Eq. (5) and Eq. (8) with the boundary conditions that come from original problem, which can be solved easily. If the series (6) convergent at $p=1$, then we have

$$\psi(s,c_i)=\psi_0(s)+\sum_{k=1}^{\infty}\psi_k(s,c_i).$$

The solution of Eq. (1) can be expressed approximately in the form:

$$\psi^m(s,c_i)=\psi_0(s)+\sum_{k=1}^{\infty}\psi_k(s,c_i),\qquad i=1,2,3\ldots.,m. \qquad (10)$$

As the solution contains the auxiliary constants c_i, $i=1,2,\ldots.$ we form the residual equation by putting Eq. (10) into Eq. (1) to find these constants as follows:

$$R(s,c_i)=L\left(\psi^m(s,c_i)\right)+g(s)+N\left(\psi^m(s,c_i)\right),\quad i=1,2,....,m. \quad (11)$$

By using the least square method, we find $c_i,\ i=1,2,....$ as

$$J(c_i)=\int_a^b R^2(s,c_i)ds. \quad (12)$$

Where a and b are the domain of the given problem. The constants $c_i, i=1,2,3,\dots,m$ can be optimally identified from the conditions

$$\frac{\partial J}{\partial c_1}=\frac{\partial J}{\partial c_2}=\frac{\partial J}{\partial c_2}=....=\frac{\partial J}{\partial c_m}=0. \quad (13)$$

With these constants, the approximate solution (of order m) in eq. (10) is well determined.

3. Implementation of OHAM to Integro-differential Equations

In this section, the effectiveness and accuracy of OHAM is tested by solving Volterra and Fredholm integro-differential equations

Problem 1

Consider the Fredhohm integro-differential equation [19] of the following form

$$u''(x)=x-\sin(x)-\int_0^{\frac{\pi}{2}} xtu(t)dt\,,\quad u(0)=0,\quad u'(0)=1\,, \quad (14)$$

Eq. (14) has exact solution $u(x)=\sin(x)$. From the given equation

$$L\left(u(x;q)\right)=u''(x)\,, \quad (15)$$

$$N\left(u(x;q)\right)=\int_0^{\frac{\pi}{2}} txu(t)dt\,, \quad (16)$$

$$f(x)=-\left(x-\sin(x)\right). \quad (17)$$

The homotopy using OHAM is as follow

$$(1-q)\left((u_0''(x)+qu_1''(x)+q^2u_2''(x)+\cdots)-\left(x-\sin(x)\right)\right)$$
$$=H(q)\begin{pmatrix}(u_0''(x)+qu_1''(x)+q^2u_2''(x)+\cdots)-\left(x-\sin(x)\right)\\ +\int_0^{\frac{\pi}{2}} tx(u_0(t)+qu_1(t)+q^2u_2(t)+\cdots)dt\end{pmatrix}, \quad (18)$$
$$u(0;q)=0,\ u'(0;q)=1$$

A series of problems is created by comparing the co-efficient of same power of q, the series is

$$O(q^0):\ u_0''(x)=x-\sin(x),\quad u_0(0)=0,\quad u_0'(0)=1\,, \quad (19)$$

$$O(q^1):\ u_1''(x)=c_1\int_0^{\frac{\pi}{2}} txu_0(x)dt\,,\ u_1(0)=0,\quad u_1'(0)=0\,, \quad (20)$$

$$O(q^2):\ u_2''(x)=\left(1+c_1\right)u_1''(x)+c_2\int_0^{\frac{\pi}{2}} txu_0(x)dt+c_1\int_0^{\frac{\pi}{2}} txu_1(x)dt,$$
$$u_2(0)=0,\quad u_2'(0)=0, \quad (21)$$

$$O(q^3):\ u_3''(x)=\left(1+c_1\right)u_2''(x)+c_2u_1''(x)+c_3\int_0^{\frac{\pi}{2}} txu_0(x)dt+c_2\int_0^{\frac{\pi}{2}} txu_1(x)dt$$
$$+c_1\int_0^{\frac{\pi}{2}} txu_2(x)dt,\ u_3(0)=0,\quad u_3'(0)=0. \quad (22)$$

Solving eq. (19-22) we have

$$u_0(x) = \frac{1}{6}\left(x^3 + 6\sin(x)\right), \tag{23}$$

$$u_1(x) = \frac{c_1\left(960+\pi^5\right)x^3}{5760}, \tag{24}$$

$$u_2(x) = \frac{1}{5529600}(960+\pi^5)(960c_1 + 960c_1^2 + 960c_2 + c_1^2\pi^5)x^3, \tag{25}$$

$$u_3(x) = \frac{1}{5308416000}\left(960+\pi^5\right)\begin{pmatrix} 921600c_1 + 1843200c_1^2 + 921600c_1^3 \\ +921600c_2 + 1843200c_1c_2 + 921600c_3 \\ +1920c_1^2\pi^5 + 1920c_1^3\pi^5 + 1920c_1c_2\pi^5 + c_1^3\pi^{10} \end{pmatrix}x^3. \tag{26}$$

The 3rd order approximate solution by OHAM is $u = u_0 + u_1 + u_2 + u_3$

$$u = \frac{1}{6}\left(x^3 + 6\sin(x)\right) + \frac{c_1\left(960+\pi^5\right)x^3}{5760} + \frac{1}{5529600}(960+\pi^5)\begin{pmatrix} 960c_1 + 960c_1^2 \\ +960c_2 + c_1^2\pi^5 \end{pmatrix}x^3$$
$$+\frac{1}{5308416000}\left(960+\pi^5\right)\begin{pmatrix} 921600c_1 + 1843200c_1^2 + 921600c_1^3 + 921600c_2 \\ +1843200c_1c_2 + 921600c_3 + 1920c_1^2\pi^5 + 1920c_1^3\pi^5 \\ +1920c_1c_2\pi^5 + c_1^3\pi^{10} \end{pmatrix}x^3. \tag{27}$$

The values of constants $c_1 = -0.89060667$, $c_2 = -1.30668415$ and $c_3 = -0.452018363$ are calculated using method of least squares.

With these constants the approximate solution (27) become

$$u = -0.1666667x^3 + \frac{1}{6}\left(x^3 + 6\sin(x)\right). \tag{28}$$

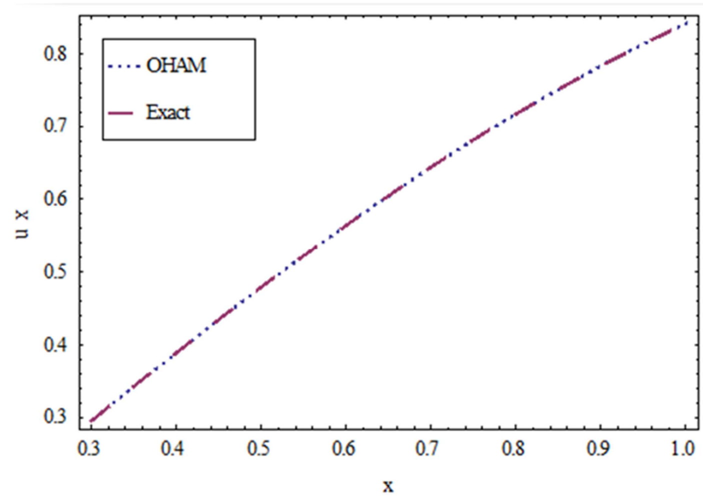

Figure 1. *2D plots of OHAM versus exact solution.*

Table 1. *Shows the comparison of 3rd order OHAM solution with 5th order ADM [19] and exact solution.*

X	Exact solutions	OHAM solutions	ADM absolute Errors	OHAM absolute Errors
0	0	0	0	0
0.1	0.099833	0.099833	3.99167×10^{-6}	0
0.2	0.198669	0.198669	3.19333×10^{-5}	0
0.3	0.295520	0.295520	1.07775×10^{-4}	5.55112×10^{-17}

X	Exact solutions	OHAM solutions	ADM absolute Errors	OHAM absolute Errors
0.4	0.389418	0.389418	2.55467×10^{-4}	0
0.5	0.479426	0.479426	4.98958×10^{-4}	5.55112×10^{-17}
0.6	0.564642	0.564642	8.62200×10^{-4}	0
0.7	0.644218	0.644218	1.36914×10^{-3}	0
0.8	0.717356	0.717356	2.04373×10^{-3}	0
0.9	0.783327	0.783327	2.90992×10^{-3}	1.11022×10^{-16}
1	0.841471	0.841471	3.99167×10^{-3}	1.11022×10^{-16}

Problem 2

Consider the following Fredhholm integro-differential equation [19]

$$u''(x) = -\sin(x) + \cos(x) + \left(2 - \frac{\pi}{2}\right)x - \int_0^{\frac{\pi}{2}} xtu(t)dt, \quad u(0) = -1, \; u'(0) = 1, \tag{29}$$

Eq. (29) has the exact solution $u(x) = \sin(x) - \cos(x)$. Using the Basic Idea of OHAM we have a series of problems

$$u_0''(x) = \left(2 - \frac{\pi}{2}\right)x + \cos(x) - \sin(x), \; u_0(0) = -1, \; u_0'(0) = 1, \tag{30}$$

$$u_1''(x) = c_1 \int_0^{\frac{\pi}{2}} txu_0(t)dt, \;\; u_1(0) = 0, \;\; u_1'(0) = 0, \tag{31}$$

$$u_2''(x) = (1 + c_1)u_1''(x) + c_2 \int_0^{\frac{\pi}{2}} txu_0(t)dt + c_1 \int_0^{\frac{\pi}{2}} txu_1(t)dt, \;\; u_2(0) = 0, \;\; u_2'(0) = 0, \tag{32}$$

$$\begin{aligned} u_3''(x) = {} & (1 + c_1)u_2''(x) + c_2 u_1''(x) + c_3 \int_0^{\frac{\pi}{2}} txu_0(t)dt + c_2 \int_0^{\frac{\pi}{2}} txu_1(t)dt \\ & + c_1 \int_0^{\frac{\pi}{2}} txu_2(t)dt, \quad u_3(0) = 0, \quad u_3'(0) = 0. \end{aligned} \tag{33}$$

Solving eq. (30 - 33) we get

$$u_0(x) = \frac{1}{12}(4x^3 - \pi x^3 - 12\cos(x) + 12\sin(x)), \tag{34}$$

$$u_1(x) = -\frac{1}{11520}\left(c_1(-3840 + 960\pi - 4\pi^5 + \pi^6)x^3\right), \tag{35}$$

$$u_2(x) = -\frac{1}{11059200}\left(960c_1 + 960c_1^2 + 960c_2 + c_1^2\pi^5\right)\left(-3840 + 960\pi - 4\pi^5 + \pi^6\right)x^3, \tag{36}$$

$$\begin{aligned} u_3(x) = {} & -\frac{1}{10616832000}\left(-3840 + 960\pi - 4\pi^5 + \pi^6\right)(921600c_1 + 1843200c_1^2 \\ & + 921600c_1^3 + 921600c_2 + 1843200c_1c_2 + 921600c_3 + 1920c_1^2\pi^5 + 1920c_1^3\pi^5 \\ & + 1920c_1c_2\pi^5 + c_1^3\pi^{10})x^3. \end{aligned} \tag{37}$$

The 3rd order approximate solution by OHAM is $u = u_0 + u_1 + u_2 + u_3$

$$\begin{aligned} u = {} & \frac{1}{12}(4x^3 - \pi x^3 - 12\cos(x) + 12\sin(x)) - \frac{1}{11520}\left(c_1(-3840 + 960\pi - 4\pi^5 + \pi^6)x^3\right) \\ & - \frac{1}{11059200}\left(960c_1 + 960c_1^2 + 960c_2 + c_1^2\pi^5\right)\left(-3840 + 960\pi - 4\pi^5 + \pi^6\right)x^3 \\ & - \frac{1}{10616832000}\left(-3840 + 960\pi - 4\pi^5 + \pi^6\right)(921600c_1 + 1843200c_1^2 \\ & + 921600c_1^3 + 921600c_2 + 1843200c_1c_2 + 921600c_3 + 1920c_1^2\pi^5 + 1920c_1^3\pi^5 \\ & + 1920c_1c_2\pi^5 + c_1^3\pi^{10})x^3. \end{aligned} \tag{38}$$

The value of constants $c_1 = -0.89060667$, $c_2 = -1.30668415$ and $c_3 = -0.45201836$ are calculated using method of least squares.

With these constants the approximate solution (38) become

$$u = -0.0715339x^3 + \frac{1}{12}(4x^3 - \pi x^3 - 12\cos(x) + 12\sin(x)) . \tag{39}$$

***Table 2.** Show the comparison of 3rd order OHAM with 4th order ADM [19] solution and exact solution.*

X	Exact solution	OHAM Solution	ADM absolute Error	OHAM absolute Errors
0	-1	-1	0	0
0.1	-0.895171	-0.895171	7.38946×10^{-7}	0
0.2	-0.781397	-0.781397	5.91156×10^{-6}	0
0.3	-0.659816	-0.659816	1.99515×10^{-5}	1.11022×10^{-16}
0.4	-0.531643	-0.531643	4.72925×10^{-5}	0
0.5	-0.398157	-0.398157	9.23682×10^{-5}	0
0.6	-0.260693	-0.260693	1.59612×10^{-4}	5.55112×10^{-17}
0.7	-0.120625	-0.120625	2.53458×10^{-4}	2.77556×10^{-17}
0.8	0.020649	0.020649	3.7834×10^{-4}	1.38778×10^{-17}
0.9	0.161717	0.161717	5.38691×10^{-4}	5.55112×10^{-17}
1	0.301169	0.301169	7.38946×10^{-4}	0

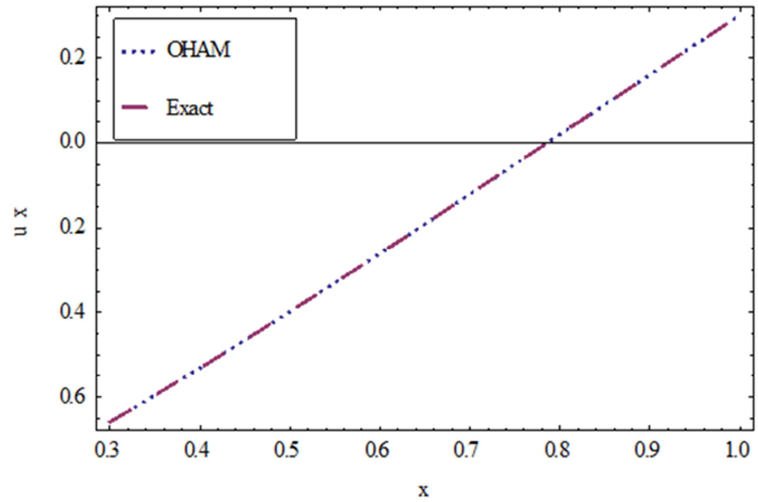

***Figure 2.** 2D plots of OHAM versus exact solution.*

Problem 3

Taking the following Fredholm integro-differential equation [19]

$$u''(x) = -e^x + \frac{x}{2} + \int_0^1 xtu(t)dt, \quad u(0) = 0, \; u'(0) = -1 , \tag{40}$$

Eq. (40) has the exact solution $u(x) = 1 - e^x$. By using Basic idea of OHAM, we have a series of problems which are as follow

$$u_0''(x) = e^x - \frac{x}{2}, \quad u_0(0) = 0, \; u_0'(0) = -1 , \tag{41}$$

$$u_1''(x) = -c_1 \int_0^1 txu_0(t)dt, \quad u_1(0) = 0, \; u_1'(0) = 0 , \tag{42}$$

$$u_2''(x) = (1 + c_1)u_1''(x) - c_2 \int_0^1 txu_0(t)dt - c_1 \int_0^1 txu_1(t)dt, \quad u_2(0) = 0, \; u_2'(0) = 0 , \tag{43}$$

$$u_3''(x) = (1+c_1)u_2''(x) + c_2u_1''(x) - c_3\int_0^1 txu_0(t)dt - c_2\int_0^1 txu_1(t)dt$$
$$-c_1\int_0^1 txu_2(t)dt,\ \ u_3(0)=0,\ \ u_3'(0)=0. \tag{44}$$

Solving Eq. (41 – 44), we have

$$u_0(x) = \frac{1}{12}\left(12 - 12e^x + x^3\right), \tag{45}$$

$$u_1(x) = \frac{29c_1x^3}{360}, \tag{46}$$

$$u_2(x) = \frac{29}{10800}(30c_1x^3 + 29c_1^2x^3 + 30c_2x^3), \tag{47}$$

$$u_3(x) = \frac{29}{324000}(900c_1x^3 + 1740c_1^2x^3 + 841c_1^3x^3 + 900c_2x^3 + 1740c_1c_2x^3 + 900c_3x^3). \tag{48}$$

The 3[rd] order approximate solution by OHAM is $u = u_0 + u_1 + u_2 + u_3$

$$u = \frac{1}{12}\left(12 - 12e^x + x^3\right) + \frac{29c_1x^3}{360} + \frac{29}{10800}(30c_1x^3 + 29c_1^2x^3 + 30c_2x^3)$$
$$+\frac{29}{324000}(900c_1x^3 + 1740c_1^2x^3 + 841c_1^3x^3 + 900c_2x^3 + 1740c_1c_2x^3 + 900c_3x^3). \tag{49}$$

The approximate solution (49) contain auxiliary constants, the values of constants $c_1 = -1.35766370$, $c_2 = -1.37973022$ and $c_3 = -0.83053608$ are calculated using method of least squares.

With these constants the approximate solution (49) become

$$u = -0.0833333x^3 + \frac{1}{12}\left(12 - 12e^x + x^3\right). \tag{50}$$

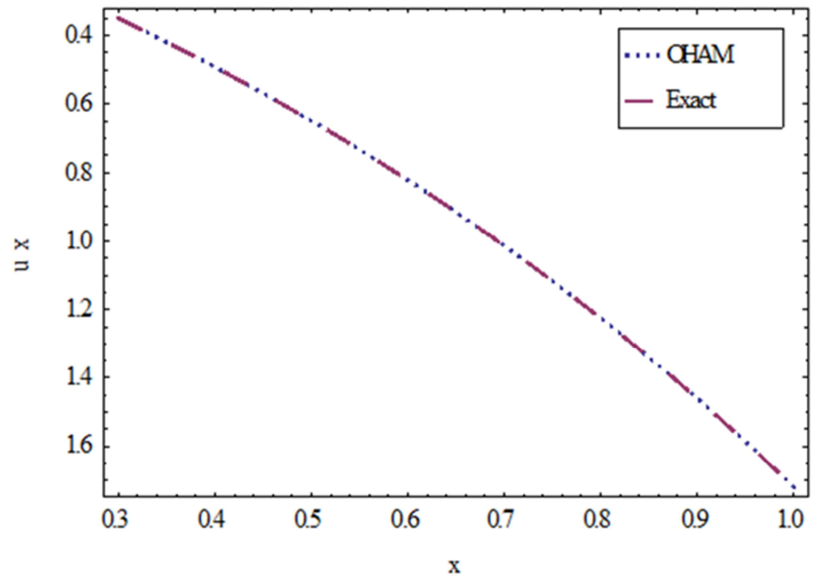

Figure 3. *2D plots of OHAM versus exact solution.*

Table 3. *Show the comparison of 3rd order OHAM solution with 4rd order ADM [19] and exact solution.*

X	Exact solutions	OHAM solutions	ADM absolute errors	OHAM absolute Errors
0	0	0	0	0
0.1	-0.10517	-0.10517	2.67490×10^{-8}	1.38778×10^{-17}
0.2	-0.22140	-0.22140	2.13992×10^{-7}	8.32667×10^{-17}
0.3	-0.34985	-0.34985	7.22222×10^{-7}	2.22045×10^{-17}
0.4	-0.49182	-0.49182	1.71193×10^{-6}	0
0.5	-0.64872	-0.64872	3.34362×10^{-6}	1.11022×10^{-17}
0.6	-0.82211	-0.82211	5.77778×10^{-6}	0
0.7	-1.01375	-1.01375	9.17490×10^{-6}	2.22045×10^{-17}
0.8	-1.22554	-1.22554	1.36955×10^{-5}	0
0.9	-1.45960	-1.45960	1.95000×10^{-5}	0
1	-1.71828	-1.71828	2.67490×10^{-5}	0

Problem 4

Consider the following Volterra integro-differential equation of the form

$$u''(x)=1+x+\int_0^x (x-t)u(t)dt,\ u(0)=1,\ u'(0)=1, \quad (51)$$

Eq. (51) has exact solution $u(x)=e^x$. By using the Basic Idea of OHAM, we have a series of problems

$$u_0''(x)=1+x, \quad u_0(0)=1,\ u_0'(0)=1, \quad (52)$$

$$u_1''(x)=-c_1\int_0^x (-t+x)u_0(t)dt, \quad u_1(0)=0,\ u_1'(0)=0, \quad (53)$$

$$u_2''(x)=(1+c_1)u_1''(x)-c_2\int_0^x (-t+x)u_0(t)dt-c_1\int_0^x (-t+x)u_1(t)dt,\ u_2(0)=0,\ u_2'(0)=0, \quad (54)$$

$$\begin{aligned} u_3''(x)=&(1+c_1)u_2''(x)+c_2u_1''(x)-c_3\int_0^x (-t+x)u_0(t)dt-c_2\int_0^x (-t+x)u_1(t)dt \\ &-c_1\int_0^x (-t+x)u_2(t)dt, \quad u_3(0)=0,\ u_3'(0)=0. \end{aligned} \quad (55)$$

Solving eqs. (52 – 55), we have

$$u_0(x)=\frac{1}{6}\left(6+6x+3x^2+x^3\right), \quad (55)$$

$$u_1(x)=\frac{1}{5040}(-210c_1x^4-42c_1x^5-7c_1x^6-c_1x^7), \quad (56)$$

$$\begin{aligned} u_2(x)=\frac{1}{39916800}&(-1663200c_1x^4-1663200c_1^2x^4-1663200c_2x^4-332640c_1x^5 \\ &-332640c_1^2x^5-332640c_2x^5-55440c_1x^6-55440c_1^2x^6-55440c_2x^6-7920c_1x^7 \\ &-7920c_1^2x^7-7920c_2x^7+990c_1^2x^8+110c_1^2x^9+11c_1^2x^{10}+c_1^2x^{11}), \end{aligned} \quad (57)$$

$$u_3(x)=\frac{1}{1307674368000}(-54486432000c_1x^4-108972864000c_1^2x^4$$
$$-54486432000c_1^3x^4-54486432000c_2x^4-108972864000c_1c_2x^4-54486432000c_3x^4$$
$$-10897286400c_1x^5-21794572800c_1^2x^5-54486432000c_3x^4-10897286400c_1x^5$$
$$-21794572800c_1^2x^5-10897286400c_3x^5-1816214400c_1x^6-3632428800c_1^2x^6$$
$$-1816214400c_1^3x^6-1816214400c_2x^6-3632428800c_1c_2x^6-1816214400c_3x^6 \quad (58)$$
$$-259459200c_1x^7-518918400c_1^2x^7-259459200c_1^3x^7-259459200c_2x^7$$
$$-518918400c_1c_2x^7-259459200c_3x^7+64864800c_1^2x^8+64864800c_1^3x^8$$
$$+64864800c_1c_2x^8+7207200c_1^2x^9+7207200c_1^3x^9+7207200c_1c_2x^9x^9$$
$$+720720c_1^2x^{10}+720720c_1^3x^{10}+720720c_1c_2x^{10}+65520c_1^2x^{11}+65520c_1^3x^{11}$$
$$+65520c_1c_2x^{11}-2730c_1^3x^{12}-210c_1^3x^{13}-15c_1^3x^{14}-c_1^3x^{15}).$$

The 3rd order approximate solution by OHAM is $u=u_0+u_1+u_2+u_3$

$$u=\frac{1}{6}\left(6+6x+3x^2+x^3\right)+\frac{1}{5040}(-210c_1x^4-42c_1x^5-7c_1x^6-c_1x^7)$$

$$+\frac{1}{39916800}(-1663200c_1x^4-1663200c_1^2x^4-1663200c_2x^4-332640c_1x^5$$
$$-332640c_1^2x^5-332640c_2x^5-55440c_1x^6-55440c_1^2x^6-55440c_2x^6-7920c_1x^7$$
$$-7920c_1^2x^7-7920c_2x^7+990c_1^2x^8+110c_1^2x^9+11c_1^2x^{10}+c_1^2x^{11})$$

$$+\frac{1}{1307674368000}(-54486432000c_1x^4-108972864000c_1^2x^4$$
$$-54486432000c_1^3x^4-54486432000c_2x^4-108972864000c_1c_2x^4-54486432000c_3x^4$$
$$-10897286400c_1x^5-21794572800c_1^2x^5-54486432000c_3x^4-10897286400c_1x^5$$
$$-21794572800c_1^2x^5-10897286400c_3x^5-1816214400c_1x^6-3632428800c_1^2x^6$$
$$-1816214400c_1^3x^6-1816214400c_2x^6-3632428800c_1c_2x^6-1816214400c_3x^6 \quad (59)$$
$$-259459200c_1x^7-518918400c_1^2x^7-259459200c_1^3x^7-259459200c_2x^7$$
$$-518918400c_1c_2x^7-259459200c_3x^7+64864800c_1^2x^8+64864800c_1^3x^8$$
$$+64864800c_1c_2x^8+7207200c_1^2x^9+7207200c_1^3x^9+7207200c_1c_2x^9x^9$$
$$+720720c_1^2x^{10}+720720c_1^3x^{10}+720720c_1c_2x^{10}+65520c_1^2x^{11}+65520c_1^3x^{11}$$
$$+65520c_1c_2x^{11}-2730c_1^3x^{12}-210c_1^3x^{13}-15c_1^3x^{14}-c_1^3x^{15}).$$

The approximate solution (59) contain auxiliary constants, the value of constants $c_1=-1.0000215$, $c_2=0$ and $c_3=0$ are calculated using method of least squares.

With these constants the approximate solution (59) become

$$u=\frac{1}{6}\left(6+6x+3x^2+x^3\right)+\frac{1}{5040}(210.0045x^4+42.0009x^5+7.0002x^6+1.00002x^7)$$
$$+\frac{1}{39916800}(-35.7551x^4-7.1510x^5-1.1918x^6-0.17026x^7+990.0426x^8$$
$$+110.0047x^9+11.0005x^{10}+1.00004x^{11})+\frac{1}{1307674368000}(25.1806x^4+5.0361x^5 \quad (60)$$
$$+0.8393x^6+0.1199x^7-1394.4790x^8-154.9421x^9-15.4942x^{10}-1.4086x^{11}$$
$$+2730.1760x^{12}+210.0135x^{13}+15.0009x^{14}+1.00006x^{15}).$$

Table 4. *Show the comparison of 3rd order OHAM solution and exact solution.*

X	Exact	OHAM	Absolute errors
0	1	1	0
0.1	1.10517	1.10517	4.44089×10^{-16}
0.2	1.22140	1.22140	0
0.3	1.34986	1.34986	0
0.4	1.49182	1.49182	0
0.5	1.64872	1.64872	2.22045×10^{-16}
0.6	1.82212	1.82212	4.44089×10^{-16}
0.7	2.01375	2.01375	4.44089×10^{-16}
0.8	2.22554	2.22554	2.22045×10^{-15}
0.9	2.45960	2.45960	1.50990×10^{-14}
1	2.71828	2.71828	5.68434×10^{-14}

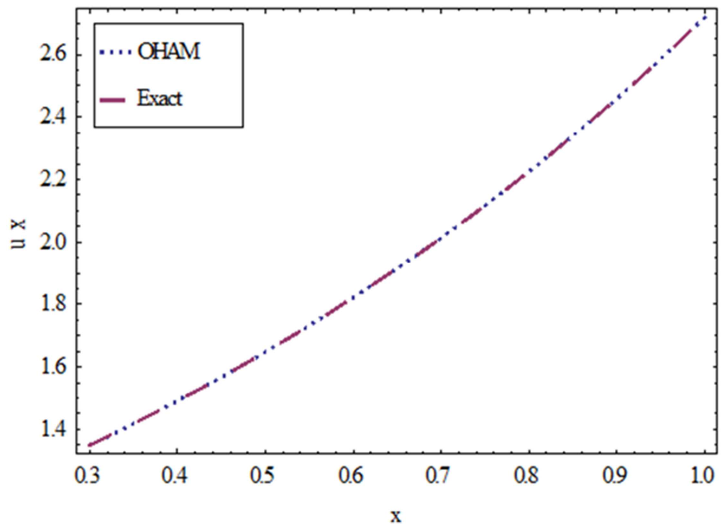

Figure 4. *2D plots of OHAM versus exact solution.*

Problem 5

Consider the following Volterra integro-differential equation of the form

$$u'(x) = 2 + \int_0^x u(t)dt, \quad u(0) = 2, \tag{61}$$

Eq. (61) has exact solution $u(x) = 2e^x$. By using the basic idea of OHAM, we have a series of problems

$$u_0'(x) = 2, \quad u_0(0) = 2, \tag{62}$$

$$u_1'(x) = -c_1 \int_0^x u_0(t)dt, \quad u_1(0) = 0, \tag{63}$$

$$u_2'(x) = (1+c_1)u_1'(x) - c_2 \int_0^x u_0(t)dt - c_1 \int_0^x u_1(t)dt, \quad u_2(0) = 0, \tag{64}$$

$$u_3'(x) = (1+c_1)u_2'(x) + c_2 u_1'(x) - c_1 \int_0^x u_2(t)dt - c_2 \int_0^x u_1(t)dt - c_3 \int_0^x u_0(t)dt, \quad u_3(0) = 0, \tag{65}$$

Solving Eqs. (62 – 65), we have

$$u_0(x) = 2(1+x), \tag{66}$$

$$u_1(x)=\frac{1}{3}\left(-3c_1x^2-c_1x^3\right), \tag{67}$$

$$u_2(x)=\frac{1}{60}\left(-60c_1x^2-60c_1^2x^2-60c_2x^2-20c_1x^3-20c_1^2x^3-20c_2x^3+5c_1^2x^4+c_1^2x^5\right), \tag{68}$$

$$\begin{aligned}u_3(x)=\frac{1}{2520}(&-2520c_1x^2-5040c_1^2x^2-2520c_1^3x^2-2520c_2x^2-5040c_1c_2x^2-2520c_3x^2\\&-840c_1x^3-1680c_1^2x^3-840c_1^3x^3-840c_2x^3-1680c_1c_2x^3-840c_3x^3+420c_1^2x^4\\&+420c_1^3x^4+420c_1c_2x^4+84c_1^2x^5+84c_1^3x^5+84c_1c_2x^5-7c_1^3x^6-c_1^3x^7).\end{aligned} \tag{69}$$

The 3rd order approximate solution by OHAM is $u=u_0+u_1+u_2+u_3$

$$\begin{aligned}u=2(1+x)&+\frac{1}{3}\left(-3c_1x^2-c_1x^3\right)+\frac{1}{60}(-60c_1x^2-60c_1^2x^2-60c_2x^2-20c_1x^3-20c_1^2x^3\\&-20c_2x^3+5c_1^2x^4+c_1^2x^5)+\frac{1}{2520}(-2520c_1x^2-5040c_1^2x^2-2520c_1^3x^2-2520c_2x^2\\&-5040c_1c_2x^2-2520c_3x^2-840c_1x^3-1680c_1^2x^3-840c_1^3x^3-840c_2x^3-1680c_1c_2x^3\\&-840c_3x^3+420c_1^2x^4+420c_1^3x^4+420c_1c_2x^4+84c_1^2x^5+84c_1^3x^5+84c_1c_2x^5-7c_1^3x^6\\&-c_1^3x^7).\end{aligned} \tag{70}$$

The estimate solution (70) contain auxiliary constants, the value of constants $c_1=-1.01277013$, $c_2=0.00018746$ and $c_3=-0.00000875$ are calculated using method of least squares.

With these constants the approximate solution (70) become

$$\begin{aligned}u=2(1+x)&+\frac{1}{3}\left(3.0383x^2+1.0128x^3\right)+\frac{1}{60}\left(-0.7872x^2-0.2624x^3+5.1285x^4+1.0257x^5\right)\\&+\frac{1}{2520}(0.9227x^2+0.3076x^3-5.5810x^4-1.1162x^5+7.2716x^6+1.0388x^7).\end{aligned} \tag{71}$$

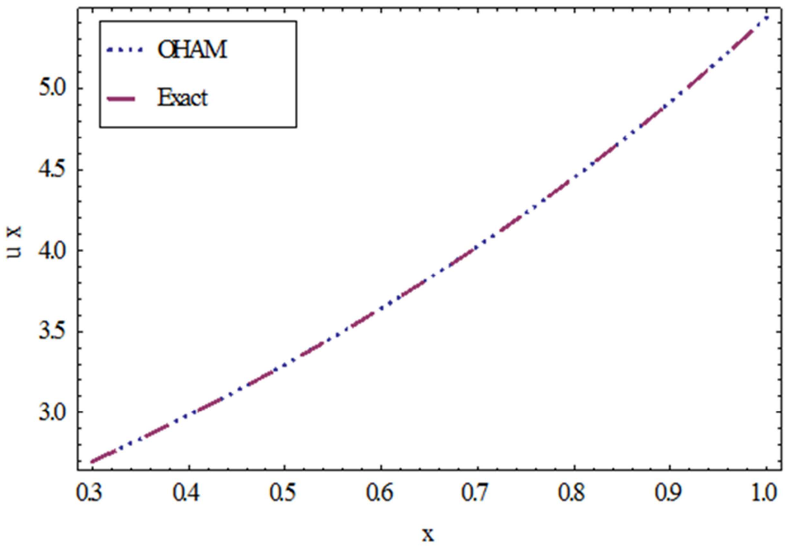

Figure 5. *2D plots of OHAM versus exact solution.*

Table 5. *Show the comparison of 3rd order OHAM solution and exact solution.*

X	Exact	OHAM	Absolute errors
0	2	2	0
0.1	2.21034	2.21034	1.54105×10^{-7}
0.2	2.44281	2.44281	5.52394×10^{-7}

X	Exact	OHAM	Absolute errors
0.3	2.69972	2.69972	1.00035×10^{-7}
0.4	2.98365	2.98365	1.25368×10^{-6}
0.5	3.29744	3.29744	1.15329×10^{-6}
0.6	3.64424	3.64424	7.54767×10^{-7}
0.7	4.02751	4.02751	3.75290×10^{-7}
0.8	4.45108	4.45108	4.51330×10^{-7}
0.9	4.91921	4.91921	1.06711×10^{-7}
1	5.43656	5.43656	9.77423×10^{-7}

4. Results and Discussions

Table 1 show the comparison of 3rd order OHAM solutions with 5th order ADM solutions for problem 1. Tables (2 – 3) show the comparison of 3rd order OHAM solutions with 4th order ADM solutions for problem (2 – 3), respectively. Tables (4 – 5) show the absolute errors of 3rd order OHAM solution for problems (4 - 5), respectively. Figures (1 – 5) show the 2D plots of exact solution versus approximate solution by OHAM for problems (1 – 5), respectively. The consistency and effectiveness of OHAM has been cleared from all these solved problems.

5. Conclusion

The intent of this attempt is to check the ability of OHAM for solving Fredholm type integro-differential equations and Volterra type integro-differential equations. OHAM gave straight forward approximate solution for these integro-differential eqautions which has close resemblance with the exist solution. From above results and discussions, it is clear that OHAM is more reliable, precise and converges faster to exact solution than ADM. The accuracy of proposed method can further be improved by taking higher order approximations.

Finally, it should be added that the presented technique has the potential to be practical in solving linear and non-linear fractional order integro-differentail equations.

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