



Functors $S^{-1}()$, $Hom_A(-, \mathcal{B})$, $- \otimes_A \mathcal{B}$ and $Ext_A^n(-, \mathcal{B})$ in Category of $A - Alg$

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Abstract: The purpose of this paper is to study some results of homological algebra in the category $A\text{-Alg}$ (resp. $\text{Alg-}A$) of left (resp. right) A -algebra in the noncommutative case. In this paper A is a subring of B . So the main results of this paper are, if B is a noetherian duo-ring, S a central saturated multiplicatively closed subset of A , S_R the set of regular elements of S , \mathcal{A} a finitely presented right A -algebra and \mathcal{B} a $(B-A)$ -bialgebra, then $Ext_A^n(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B$ is isomorphic to $S_R^{-1}Ext_A^n(\mathcal{A}, \mathcal{B})$, also $S_R^{-1}Ext_A^n(\mathcal{A}, \mathcal{B})$ is isomorphic to $Ext_{S^{-1}A}^n(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$, for any integer n .

Keywords: Algebra, Multiplicatively Closed Subset, Ore Condition of Multiplicatively Closed Subset, Localized, Category, Functor, Complex Projective Resolution

1. Introduction

In this paper, A is assumed unitary, associative and not necessarily commutative. \mathcal{A} and \mathcal{B} are algebras assumed unitary, associative and not necessarily commutative as a ring and unital as a A -module.

In general, the action of the functors $Hom_A(-, \mathcal{B})$, $- \otimes_A \mathcal{B}$ and $Ext_A^n(-, \mathcal{B})$ on a A -module M (resp. A -algebra \mathcal{A}) is not an algebra.

In this paper the conditions in which $Hom_A(M, \mathcal{B})$ and $Ext_A^n(M, \mathcal{B})$ have a structure of algebra are given. The conditions in which $Hom_A(\mathcal{A}, \mathcal{B})$, $\mathcal{A} \otimes_A \mathcal{B}$ and $Ext_A^n(\mathcal{A}, \mathcal{B})$ have a structure of algebra are also given, and the localization of its algebras with a multiplicatively closed subsets satisfying the left Ore condition is studied.

The main purpose of this paper is to study the functorial relations between the functor localization $S^{-1}()$ with the

functors $Hom_A(-, \mathcal{B}): Alg - A \rightarrow B - Alg$, $- \otimes_A \mathcal{B}: Alg - B \rightarrow Alg - A$ and $Ext_A^n(-, \mathcal{B}): Alg - A \rightarrow B - Alg$.

In this paper the following main results are shown:

In the section 3:

If \mathcal{A} is a left (resp. right) A -algebra and S a central saturated multiplicatively closed subset of A , then the left (resp. right) $S^{-1}A$ -module $S^{-1}\mathcal{A}$ is a left (resp. right) $S^{-1}A$ -algebra, furthermore $\mathcal{A} \otimes_A S^{-1}A \cong S^{-1}\mathcal{A}$.

If B is a duo-ring, A a subring of B , S a central saturated multiplicatively closed subset of A , S_R the set of regular elements of S , \mathcal{A} right A -algebra and \mathcal{B} a $(B - A)$ -bialgebra then $Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is a left $S_R^{-1}B$ -algebra.

If B is a duo-ring, A a subring of B , S a central saturated multiplicatively closed subset of A , S_R the set of regular elements of S , \mathcal{A} a finitely presented right A -algebra and \mathcal{B} a $(B - A)$ -bialgebra, then there exists the isomorphisms of left $S_R^{-1}B$ -algebras,

$$Hom_A(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Hom_A(\mathcal{A}, \mathcal{B}) \cong Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

furthermore if \mathcal{P} is a prime ideal of B , $S = (B - \mathcal{P}) \cap Z(B)$ and S_R the set of regular elements of S , then there exists the isomorphisms of left $S^{-1}B$ -algebras,

$$Hom_B(\mathcal{A}, \mathcal{B}) \otimes_B S^{-1}B \cong S^{-1}Hom_B(\mathcal{A}, \mathcal{B}) \cong Hom_{S^{-1}B}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

And furthermore also there exists the isomorphisms of left $S_R^{-1}B$ -algebras,

$$Hom_B(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Hom_B(\mathcal{A}, \mathcal{B}) \cong Hom_{S^{-1}B}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

If B is a ring, S a saturated multiplicatively closed subset of $Z(B)$, \mathcal{A} a finitely presented right $Z(B)$ -algebra and \mathcal{B} a $(B - Z(B))$ -bialgebra, then there exists an isomorphisms of left $S^{-1}B$ -algebras,

$$Hom_{Z(B)}(\mathcal{A}, \mathcal{B}) \otimes_B S^{-1}B \cong S^{-1}Hom_{Z(B)}(\mathcal{A}, \mathcal{B}) \cong Hom_{S^{-1}Z(B)}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

Furthermore if \mathcal{P} a prime ideal of $Z(B)$, then it exists the isomorphisms of left $Z(B)_{\mathcal{P}}$ -algebras,

$$Hom_{Z(B)}(\mathcal{A}, \mathcal{B}) \otimes_B B_{\mathcal{P}} \cong Hom_{Z(B)}(\mathcal{A}, \mathcal{B})_{\mathcal{P}} \cong Hom_{Z(B)_{\mathcal{P}}}(\mathcal{A}_{\mathcal{P}}, \mathcal{B}_{\mathcal{P}}).$$

In the section 4:

If \mathcal{A} is a right A -algebra, \mathcal{B} a $(B - A)$ -bialgebra the unitary left B -algebra, $Ext_A^n(\mathcal{A}, \mathcal{B})$ is built as follows:

if $P: P_n \xrightarrow{d_n} P_{n-1} \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathcal{A} \rightarrow 0$ is a complex projective resolution of a right A -module \mathcal{A} , then

$$Ext_A^n(\mathcal{A}, \mathcal{B}) = [Ker d_{n+1}^*] / \langle Im d_n^* \rangle,$$

where

$$d_n^* = Hom_A(d_n, \mathcal{B}): Hom_A(P_{n-1}, \mathcal{B}) \rightarrow Hom_A(P_n, \mathcal{B})$$

$$Ext_A^n(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Ext_A^n(\mathcal{A}, \mathcal{B}) \cong Ext_{S^{-1}A}^n(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

2. Preliminary Results

Definition 2.1

(1). A subset S of a ring A is called multiplicative if $1_A \in S$ and S is stable by multiplication i.e for all $x, t \in S$, $xt \in S$.

(2). A multiplicative subset S of a ring A is called closed if for all $s, s' \in A$ such that $ss' \in S \Rightarrow s \in S$ and $s' \in S$.

(3). Let S a multiplicatively closed subset of A . S satisfies the left Ore conditions.

If :

(a) $\forall a \in A, \forall s \in S \exists t \in S$ and $b \in A$ such that $ta = bs$

(b) $\forall a \in A, \forall s \in S$ such that $sa = 0$, then it exists $t \in S$ such that $at = 0$.

Theorem 2.1 Let A a ring, M a left A -module and S a multiplicatively closed subset of A satisfying the left Ore conditions. The binary relation defined in $S \times M$ by

$$(s, m)\mathcal{R}(s', m') \Leftrightarrow \exists x, y \in S / \begin{cases} xm = ym' \\ xx = ys' \end{cases}$$

is an equivalence relation.

Proof

See [14] ■

Notation: $S^{-1}M$ is noted the set of equivalence classes modulo \mathcal{R} ($S \times M/\mathcal{R}$).

If $(s, m) \in S \times M$, then the classe $\overline{(s, m)}$ is noted $\frac{m}{s}$, in particular if $M = A$ the classe $\overline{(s, a)} \in S^{-1}A$ is noted by $\frac{a}{s}$.

Theorem 2.2 Let A a ring not necessary commutative, M a

$$\varphi_n \mapsto \varphi_n \circ d_n,$$

$[Ker d_{n+1}^*]$ is the unitary B -algebra generated by $Ker d_{n+1}^*$ and $\langle Im d_n^* \rangle$ is the ideal of $[Ker d_{n+1}^*]$ generated by $Im d_n^*$.

If B is a noetherian duo-ring, A a subring of B , S a central saturated multiplicatively closed subset of A , S_R the set of regular elements of S , \mathcal{A} a finitely generated right A -algebra and \mathcal{B} a $(B - A)$ -bialgebra, then it exists the isomorphisms of left $S_R^{-1}B$ -algebras,

left A -module and S a multiplicatively closed subset of A satisfying the left Ore conditions, $S^{-1}A$ is a ring by the two following operations:

1. $\frac{a}{t} + \frac{b}{s} = \frac{xa+yb}{ys}$, where $x, y \in S: xt = ys$
2. $\frac{a}{t} \times \frac{b}{s} = \frac{zb}{wt}$, where $(w, z) \in S \times A: wa = zs$.

$S^{-1}M$ is a left $S^{-1}A$ -module by the two following operations:

1. $\frac{m}{s} + \frac{m'}{s'} = \frac{xm+ym'}{ys}$, where $x, y \in S$ such that $xs = ys$.
2. $\frac{a}{t} \cdot \frac{m}{s} = \frac{zm}{wt}$, where $(w, z) \in S \times A$ such that $wa = zs$.

Proof

See [13] and [14] ■

Proposition 2.1 Let A and \mathcal{A} two rings, and $\theta: A \rightarrow \mathcal{A}$ a morphism of rings. Then \mathcal{A} has a structure of left (resp. right) A -module as follows:

$$\bullet: A \times \mathcal{A} \rightarrow \mathcal{A}$$

$$(a, x) \mapsto a \bullet x = \theta(a)x$$

$$\text{(resp. } *: \mathcal{A} \times A \rightarrow \mathcal{A}$$

$$(x, a) \mapsto x * a = x\theta(a))$$

Proof

Easy. ■

Definition 2.2 Let A and \mathcal{A} two rings, and $\theta: A \rightarrow \mathcal{A}$ a morphism of rings. Then $(\mathcal{A}, +, \times, \bullet)$ (resp. $(\mathcal{A}, +, \times, *)$) is called a left (resp. right) A -algebra relatively to θ .

Remark 1 This definition shows that to provide \mathcal{A} with a structure of left (or right) A -algebra it is enough to have a morphism of rings from A to \mathcal{A} .

Definition 2.3 Let A and \mathcal{A} two rings, and $\theta: A \rightarrow \mathcal{A}$ a morphism of rings. If $Im(\theta) \subseteq Z(\mathcal{A})$, then \mathcal{A} is called a A -algebra relatively to θ .

Proposition 2.2 Let A and \mathcal{A} two rings.

(1). If \mathcal{A} is a left A -module such that $a \cdot (xy) = (a \cdot x)y = x(a \cdot y)$, $\forall a \in A, \forall x, y \in \mathcal{A}$, then \mathcal{A} is a left A -algebra.

(2). If \mathcal{A} is a right A -module such that $(xy) \cdot a = x(y \cdot a) = (x \cdot a)y$, $\forall a \in A, \forall x, y \in \mathcal{A}$, then \mathcal{A} is a right A -algebra.

Proof

* Suppose that \mathcal{A} is a left A -module such that $a \cdot (xy) = (a \cdot x)y = x(a \cdot y)$, $\forall a \in A, \forall x, y \in \mathcal{A}$ and show that \mathcal{A} is a left A -algebra.

Consider the canonical morphism of left A -modules:

$$\theta_g: A \rightarrow \mathcal{A}$$

$$a \mapsto a \cdot 1_{\mathcal{A}}.$$

It is enough to show that θ_g is an isomorphism of rings.

It is clear that θ_g is a morphism of groups.

Let $a_1, a_2 \in A$ we have:

$$\theta_d(a_1 a_2) = 1_{\mathcal{A}} \cdot (a_1 a_2) = (1_{\mathcal{A}} \cdot a_1) \cdot a_2 = [(1_{\mathcal{A}} \cdot a_1) 1_{\mathcal{A}}] \cdot a_2 = (1_{\mathcal{A}} \cdot a_1)(1_{\mathcal{A}} \cdot a_2) = \theta_d(a_1) \theta_d(a_2).$$

So θ_d is a morphism of rings, so \mathcal{A} is a right A -algebra. ■

3. Functor $S^{-1}()$ and Functor $Hom_A(-, \mathcal{B})$ in the Category $A\text{-Alg}$ (resp. $\text{Alg-}A$)

3.1. Algebra of Fraction in the Non Commutative Case

Theorem 3.1 Let \mathcal{A} a left (resp. right) A -algebra and S a central saturated multiplicatively closed subset of A . Then the left (resp. right) $S^{-1}A$ -module, $S^{-1}\mathcal{A}$ is a left (resp. right) $S^{-1}A$ -algebra.

Proof

Show that $S^{-1}\mathcal{A}$ is a left $S^{-1}A$ -algebra.

Define a law of internal composition on $S^{-1}\mathcal{A}$.

Consider the following correspondence:

$$\tilde{\times}: S^{-1}\mathcal{A} \times S^{-1}\mathcal{A} \rightarrow S^{-1}\mathcal{A}$$

$$\left(\frac{x}{t}, \frac{x'}{t'}\right) \mapsto \frac{x}{t} \tilde{\times} \frac{x'}{t'} = \frac{xx'}{tt'}.$$

Let $\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right), \left(\frac{x'_1}{t'_1}, \frac{x'_2}{t'_2}\right) \in S^{-1}\mathcal{A} \times S^{-1}\mathcal{A}$ such that $\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) = \left(\frac{x'_1}{t'_1}, \frac{x'_2}{t'_2}\right)$.

So it exists $a, b, a', b' \in S$ such that

$$\theta_g(a_1 a_2) = (a_1 a_2) \cdot 1_{\mathcal{A}} = a_1 \cdot (a_2 \cdot 1_{\mathcal{A}})$$

$$= a_1 \cdot [(a_2 \cdot 1_{\mathcal{A}}) 1_{\mathcal{A}}] = a_1 \cdot [1_{\mathcal{A}}(a_2 \cdot 1_{\mathcal{A}})]$$

$$= (a_1 1_{\mathcal{A}})(a_2 \cdot 1_{\mathcal{A}}) = \theta_g(a_1) \theta_g(a_2).$$

So θ_g is a morphism of rings, so $(\mathcal{A}, +, \times, \bullet)$ is a left A -algebra where "•" the law induced by θ_g .

* Suppose that \mathcal{A} is a left A -module such that $(xy) \cdot a = x(y \cdot a)$, $\forall a \in A, \forall x, y \in \mathcal{A}$ and show that \mathcal{A} is a left A -algebra.

Consider the canonical morphism of left A -modules:

$$\theta_d: A \rightarrow \mathcal{A}$$

$$a \mapsto 1_{\mathcal{A}} \cdot a.$$

It is enough to show that θ_d is a morphism of rings.

It is clear that θ_d is a morphism of groups.

Let $a_1, a_2 \in A$

Then:

$$\begin{cases} \begin{cases} a \cdot x_1 = b \cdot x'_1 & (1) \\ at_1 = bt'_1 & (2) \end{cases} \\ \begin{cases} a' \cdot x_2 = b' \cdot x'_2 & (1)' \\ a't_2 = b't'_2 & (2)' \end{cases} \end{cases}$$

Whereas:

$$\begin{cases} \frac{x_1}{t_1} \tilde{\times} \frac{x_2}{t_2} = \frac{x_1 x_2}{t_1 t_2} \\ \frac{x'_1}{t'_1} \tilde{\times} \frac{x'_2}{t'_2} = \frac{x'_1 x'_2}{t'_1 t'_2} \end{cases}$$

Whereas:

$$(a \cdot x_1)(a' \cdot x_2) = a \cdot [x_1(a' \cdot x_2)] \text{ because } \mathcal{A} \text{ is a left } A\text{-algebra.}$$

$$= x_1(a \cdot (a' \cdot x_2)) \text{ because } \mathcal{A} \text{ is a left } A\text{-algebra}$$

$$= x_1((aa') \cdot x_2) \text{ because } \mathcal{A} \text{ is a } A\text{-module}$$

$$= (aa') \cdot (x_1 x_2) \text{ because } \mathcal{A} \text{ is a left } A\text{-algebra.}$$

Whereas also:

$$(b \cdot x'_1)(b' \cdot x'_2) = b \cdot [x'_1(b' \cdot x'_2)] \text{ because } \mathcal{A} \text{ is a left } A\text{-algebra}$$

$= x'_1(b \cdot (b' \cdot x'_2))$ because \mathcal{A} is a left A -algebra

$= x'_1((bb') \cdot x'_2)$ because \mathcal{A} is a left A -module

$= (bb') \cdot (x'_1 x'_2)$ because \mathcal{A} is a left A -algebra.

Or (1) and (1)' $\Rightarrow (a \cdot x_1)(a' \cdot x_2) = (b \cdot x'_1)(b' \cdot x'_2)$.

So $(aa') \cdot (x_1 x_2) = (bb') \cdot (x'_1 x'_2)$ (*).

Whereas: $(at_1)(a't_2) = (aa')(t_1 t_2)$ because $S \subset Z(A)$.

Similarly we have $(bt'_1)(b't'_2) = (bb')(t'_1 t'_2)$ because $S \subset Z(A)$

(2) and (2)' $\Rightarrow (at_1)(a't_2) = (bt'_1)(b't'_2) \Rightarrow (aa')(t_1 t_2) = (bb')(t'_1 t'_2)$ (**).

Pose $X = aa'$ and $Y = bb'$.

Whereas

$$\begin{cases} X(x_1 x_2) = Y(x'_1 x'_2) \\ X(t_1 t_2) = Y(t'_1 t'_2) \end{cases}$$

So

$$\frac{x_1}{t_1} \tilde{\times} \frac{x_2}{t_2} = \frac{x'_1}{t'_1} \tilde{\times} \frac{x'_2}{t'_2}.$$

Therefore $\tilde{\times}$ is a law internal composition.

After [14] $S^{-1}\mathcal{A}$ is a left $S^{-1}A$ -module, so to show that $S^{-1}\mathcal{A}$ is a left $S^{-1}A$ -algebra, it is enough to show after the proposition 2.2 that:

$$\forall \frac{a}{s} \in S^{-1}A, \forall \frac{x}{t}, \frac{x'}{t'} \in S^{-1}\mathcal{A}, \frac{a}{s} \cdot \left(\frac{x}{t} \tilde{\times} \frac{x'}{t'} \right) = \frac{x}{t} \tilde{\times} \left(\frac{a}{s} \cdot \frac{x'}{t'} \right) = \left(\frac{a}{s} \cdot \frac{x}{t} \right) \tilde{\times} \frac{x'}{t'}.$$

Whereas first:

$$\frac{a}{s} \cdot \left(\frac{x}{t} \tilde{\times} \frac{x'}{t'} \right) = \frac{a}{s} \cdot \frac{xx'}{tt'} = \frac{z \cdot (xx')}{ws} \text{ where } (w, z) \in S \times A \text{ such that } wa = zt t' (E_1).$$

Whereas second:

$$\frac{x}{t} \tilde{\times} \left(\frac{a}{s} \cdot \frac{x'}{t'} \right) = \frac{x}{t} \tilde{\times} \frac{z'x'}{w's} = \frac{xx'z'}{tw's} \text{ where } (w', z') \in S \times A \text{ such that } w'a = z't' (E_2).$$

Since $(w, tw') \in S \times S \exists (p, q) \in S \times S$ such that $pw = qtw' \Rightarrow pwa = qtw'a$, so after (E_1) et (E_2) $pzt t' = qtz' t' \Rightarrow pzt t' - qtz' t' = 0 \Rightarrow (pzt - qtz')t' = 0$,

so after the seconde Ore condition $\exists r_1 \in S$ such that $r_1(pzt - qtz') = 0 \Rightarrow r_1 pzt = r_1 qtz' \Rightarrow r_1 pzt \cdot (xx') = r_1 qtz' \cdot (xx')$.

Calculate first $r_1 pzt \cdot (xx')$:

Whereas:

$(r_1 pzt) \cdot (xx') = (r_1 ptz) \cdot (xx')$ because $zt = tz$ since $S \subset Z(A) = ((r_1 pt)z) \cdot (xx') = (r_1 pt) \cdot (z \cdot (xx'))$ because \mathcal{A} is a left A -module.

Then calculate $[(r_1 qt)z'] \cdot (xx')$

Whereas:

$[(r_1 qt)z'] \cdot (xx') = (r_1 qt) \cdot (z' \cdot (xx'))$ because \mathcal{A} is a left A -module $= (r_1 qt) \cdot [x(z' \cdot x)]$ because \mathcal{A} is a left

A -algebra.

Also since $pw = qtw'$ then $r_1 tpw = r_1 tqtw'$ and since $S \subset Z(A)$ then $(r_1 pt)(ws) = (r_1 qt)(tw')$.

Pose $X = r_1 pt$ and $Y = r_1 qt$

Whereas:

$$\begin{cases} X(z \cdot (xx')) = Y(x(z' \cdot x')) \\ X(ws) = Y(tw') \end{cases} \Leftrightarrow \frac{z \cdot (xx')}{ws} = \frac{x \cdot (z'a)}{wt't}.$$

So

$$\frac{a}{s} \cdot \left(\frac{x}{t} \tilde{\times} \frac{x'}{t'} \right) = \frac{x}{t} \tilde{\times} \left(\frac{a}{s} \cdot \frac{x'}{t'} \right).$$

In the same way we show that $\frac{a}{s} \cdot \left(\frac{x}{t} \tilde{\times} \frac{x'}{t'} \right) = \left(\frac{a}{s} \cdot \frac{x}{t} \right) \tilde{\times} \frac{x'}{t'}$.

So

$$\frac{a}{s} \cdot \left(\frac{x}{t} \tilde{\times} \frac{x'}{t'} \right) = \left(\frac{a}{s} \cdot \frac{x}{t} \right) \tilde{\times} \frac{x'}{t'} = \frac{x}{t} \tilde{\times} \left(\frac{a}{s} \cdot \frac{x'}{t'} \right), \forall \frac{a}{s} \in S^{-1}A, \forall \frac{x}{t}, \frac{x'}{t'} \in S^{-1}\mathcal{A}.$$

Therefore $S^{-1}\mathcal{A}$ is a left $S^{-1}A$ -algebra. ■

Proposition 3.1 Let B and A two rings.

(1). If M is a right A -module and \mathcal{B} is a $(B - A)$ -bialgebra, then the group $Hom_A(M, \mathcal{B})$ is a unitary left B -algebra by the following operations:

$$(b \cdot f)(m) = bf(m), \forall m \in M, \forall f \in Hom_A(M, \mathcal{B}), \forall b \in B,$$

$$(f g)(m) = f(m)g(m), \forall m \in M, \forall f, g \in Hom_A(M, \mathcal{B}).$$

(2). If M is a left A -module and \mathcal{B} is a $(A - A)$ -bialgebra, then the group $Hom_A(M, \mathcal{B})$ is a unitary left A -algebra by the following operations:

$$(a \cdot f)(m) = af(m), \forall m \in M, \forall f \in Hom_A(M, \mathcal{B}), \forall a \in A,$$

$$(f g)(m) = f(m)g(m), \forall m \in \mathcal{A}, \forall f, g \in Hom_A(M, \mathcal{B}).$$

Proof

- the group $Hom_A(M, \mathcal{B})$ is a unitary left B -algebra.
- * After [18] $Hom_A(M, \mathcal{B})$ is a left B -module.
- * Show that $Hom_A(M, \mathcal{B})$ has a structure of ring via the operation defined above.

Let $f, g, h \in Hom_A(M, \mathcal{B})$, $m \in M$, we have:

$$\begin{aligned} [(fg)h](m) &= (fg)(m)h(m) = (f(m)g(m))h(m) \\ &= f(m)(g(m)h(m)), \text{ because } \mathcal{B} \text{ is a ring} \\ &= f(m)(gh)(m) = [f(gh)](m). \end{aligned}$$

So $(fg)h = f(gh)$, so the internal composition law is associative.

Whereas also:

$$\begin{aligned} [f(g+h)](m) &= f(m)(g+h)(m) \\ &= f(m)(g(m) + h(m)) \end{aligned}$$

$$= f(m)g(m) + f(m)h(m) = (fg + fh)(m).$$

So $f(g + h) = fg + fh$, so the internal composition law is distributive compared with the addition.

Therefore, $Hom_A(M, \mathcal{B})$ has a structure of ring of unit element,

$$1: M \rightarrow \mathcal{B}.$$

$$m \mapsto 1_{\mathcal{B}}$$

* Show that the two laws are compatible

Let $b \in B, f, g \in Hom_A(M, \mathcal{B}), m \in M$

$$\text{whereas } [b \cdot (fg)](m) = b(fg)(m) = b(f(m)g(m)) = (bf(m))g(m)$$

$$\text{because } \mathcal{B} \text{ is a left } B\text{-algebra } = (b \cdot f)(m)g(m) = [(b \cdot f)g](m)$$

$$\text{whereas also } [b \cdot (fg)](m) = b(fg)(m) = b(f(m)g(m))$$

$$= f(m)(bg(m)) \text{ because } \mathcal{B} \text{ is a left } B\text{-algebra.}$$

$$= f(m)(b \cdot g)(m) = [f(b \cdot g)](m).$$

$$\text{So } b \cdot (fg) = (b \cdot f)g = f(b \cdot g).$$

So after the proposition 2.2 $Hom_A(M, \mathcal{B})$ is a left B -algebra.

• The item 2. is showing the same way. ■

Corollary 3.1

(1). If \mathcal{A} is a right A -algebra and \mathcal{B} is a $(B - A)$ -bialgebra, then the group $Hom_A(\mathcal{A}, \mathcal{B})$ is a unitary left B -algebra by the following operations:

$$(b \cdot f)(x) = bf(x), \forall x \in \mathcal{A}, \forall f \in Hom_A(\mathcal{A}, \mathcal{B}), \forall b \in B,$$

$$(f g)(x) = f(x)g(x), \forall x \in \mathcal{A}, \forall f, g \in Hom_A(\mathcal{A}, \mathcal{B}).$$

(2). If \mathcal{A} is a left A -algebra and \mathcal{B} is a $(A - A)$ -bialgebra, then group $Hom_A(\mathcal{A}, \mathcal{B})$ is a left A -algebra by the following operations:

$$(a \cdot f)(x) = af(x), \forall x \in \mathcal{A}, \forall f \in Hom_A(\mathcal{A}, \mathcal{B}), \forall a \in A,$$

$$(f g)(x) = f(x)g(x), \forall f, g \in Hom_A(\mathcal{A}, \mathcal{B}), x \in \mathcal{A}.$$

Proof

Just consider the A -algebra \mathcal{A} as a A -module and you have the results 1 and 2. ■

Proposition 3.2 Let B and A two rings.

(1). If \mathcal{A} is a right B -algebra and \mathcal{B} is a $(B - A)$ -bialgebra, then the group $\mathcal{A} \otimes_B \mathcal{B}$ is a right A -algebra of unit element $1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$ by the following operations:

$$(x \otimes y).a = x \otimes (y.a), \forall x \in \mathcal{A}, \forall y \in \mathcal{B}, \forall a \in A$$

$$(x \otimes y)(x' \otimes y') = (xx' \otimes yy'), \forall x, x' \in \mathcal{A}, \forall y, y' \in \mathcal{B}.$$

(2). If \mathcal{A} is a left A -algebra and \mathcal{B} is a $(B - A)$ -bialgebra, then the group $\mathcal{A} \otimes_A \mathcal{B}$ is a unitary left B -algebra by the following operations:

$$b \cdot (x \otimes y) = x \otimes (b.y), \forall x \in \mathcal{A}, \forall y \in \mathcal{B}, \forall b \in B,$$

$$(x \otimes y)(x' \otimes y') = (xx' \otimes yy'), \forall x, x' \in \mathcal{A}, \forall y, y' \in \mathcal{B}.$$

Proof

• $\mathcal{A} \otimes_B \mathcal{B}$ is a right A -algebra.

* After [1] $\mathcal{A} \otimes_B \mathcal{B}$ is a right A -module.

* It's not difficult to show that $\mathcal{A} \otimes_B \mathcal{B}$ has a structure of ring.

* Show that the two laws are compatible.

Let $x \otimes y, x' \otimes y' \in \mathcal{A} \otimes_B \mathcal{B}$ and $a \in A$.

whereas $[(x \otimes y)(x' \otimes y')].a = (xx' \otimes yy').a = (xx') \otimes ((yy').a) = (xx') \otimes (y(y'.a))$ because \mathcal{B} is a right A -algebra. $= (x \otimes y)(x' \otimes y'.a) = (x \otimes y)[(x' \otimes y').a]$ whereas $[(x \otimes y)(x' \otimes y')].a = (xx' \otimes yy').a = (xx') \otimes ((yy').a) = (xx') \otimes ((y.a)y')$ because \mathcal{B} is a right A -algebra. $= (x \otimes (y.a))(x' \otimes y') = [(x \otimes y).a](x' \otimes y')$

$$\text{So } [(x \otimes y)(x' \otimes y')].a = (x \otimes y)[(x' \otimes y').a] = [(x \otimes y).a](x' \otimes y').$$

So after the proposition 2.2 $\mathcal{A} \otimes_B \mathcal{B}$ is a right A -algebra.

• The item 2. is showing the same way. ■

Theorem 3.2 Let B a duo-ring, A a subring of B, \mathcal{B} a left B -algebra relatively to θ, S a central saturated multiplicatively closed subset of A and S_R the set of regular elements of S . Then $S^{-1}\mathcal{B}$ is a left $S_R^{-1}B$ -algebra.

Proof

After the theorem 3.1 $S^{-1}\mathcal{B}$ is a left $S^{-1}A$ -algebra $\Rightarrow S^{-1}\mathcal{B}$ is a ring.

Since S_R is the set of regular elements of the duo-ring B , so S_R is a saturated multiplicatively closed subset of B satisfying the Ore conditions, so after [13] $S_R^{-1}B$ is also a ring.

So it is enough to show that this following correspondence is a morphism of rings:

$$\theta': S_R^{-1}B \rightarrow S^{-1}\mathcal{B}$$

$$\frac{b}{s} \mapsto \frac{\theta(b)}{s}$$

So after the remark 1 $S^{-1}\mathcal{B}$ is a left $S_R^{-1}B$ -algebra. ■

Theorem 3.3 Let B a ring, \mathcal{B} a left B -algebra relatively to θ and S a saturated multiplicatively closed subset of $Z(B)$. Then $S^{-1}\mathcal{B}$ is a left $S^{-1}B$ -algebra.

Proof

The proof is similar to the previous one. ■

Theorem 3.4 Let B a duo-ring, A a subring of B, S a central saturated multiplicatively closed subset of A, S_R the set of regular elements of S, \mathcal{A} right A -algebra and \mathcal{B} a $(B - A)$ -bialgebra. Then $Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is a left $S_R^{-1}B$ -algebra,

Proof

After the theorem 3.1, $S^{-1}\mathcal{A}$ is a right $S^{-1}A$ -algebra and $S^{-1}\mathcal{B}$ is a right $S^{-1}A$ -algebra.

After the theorem 3.2, $S^{-1}\mathcal{B}$ has a structure of left $S_R^{-1}B$ -algebra.

So $S^{-1}\mathcal{B}$ is a $(S_R^{-1}B - S^{-1}A)$ -bialgebra, so after the proposition 3.1, $Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ has a structure of left $S_R^{-1}B$ -algebra. ■

3.2. Relationship Between Functors $S^{-1}()$, $Hom_A(-, \mathcal{B})$ and $- \otimes \mathcal{B}$ in the Category $A - Alg$ (resp. $Alg - A$)

Proposition 3.3 Let \mathcal{B} a $(B - A)$ -bialgebra. Then the

correspondence

$$Hom_A(-, \mathcal{B}): Mod - A \rightarrow B - Alg$$

(1). who has any right A -module M , we associate the left B -algebra $Hom_A(M, \mathcal{B})$,

(2). who has any morphism of right A -modules $f: M \rightarrow M'$, we associate $f^*Hom_A(f, \mathcal{B}): Hom_A(M', \mathcal{B}) \rightarrow Hom_A(M, \mathcal{B})$ is a contravariant functor.

Proof

* After the proposition 3.1 $M \in Ob(Mod - A) \Rightarrow Hom_A(M, \mathcal{B}) \in Ob(B - Alg)$.

* Let $f: M \rightarrow M'$ a morphism of right A -modules, show that

$f^* = Hom_A(f, \mathcal{B}): Hom_A(M', \mathcal{B}) \rightarrow Hom_A(M, \mathcal{B})$ is a morphism of left B -algebras..

Let $\varphi, \psi \in Hom_A(M, \mathcal{B})$, $x \in M$.

We have

$$\begin{aligned} [f^*(\varphi\psi)](x) &= [(\varphi\psi) \circ f](x) = \varphi(f(x))\psi(f(x)) \\ &= (\varphi \circ f)(x)(\psi \circ f)(x) \\ &= [(\varphi \circ f)(\psi \circ f)](x) \\ &= [f^*(\varphi)f^*(\psi)](x). \end{aligned}$$

So we have $f^*(\varphi\psi) = f^*(\varphi)f^*(\psi)$.

Let $\varphi \in Hom_A(M, \mathcal{B})$, $b \in \mathcal{B}$, $x \in M$.

We have

$$\begin{aligned} [f^*(b\varphi)](x) &= [(b\varphi) \circ f](x) = (b\varphi)(f(x)) = b(\varphi \circ f)(x) \\ &= [b(f^*(\varphi))](x). \end{aligned}$$

So we have $f^*(b\varphi) = b(f^*(\varphi))$.

Therefore f^* , for all $n \geq 0$, is a morphism of left B -algebras.

* Let $f: M \rightarrow M'$ and $g: M' \rightarrow M''$ two morphism of right A -modules, $x \in M$, we have:

$$\begin{aligned} Hom_A(g \circ f, \mathcal{B})(\varphi)(x) &= [\varphi \circ (g \circ f)](x) \\ &= [Hom_A(f, \mathcal{B}) \circ (\varphi \circ g)](x) \\ &= [Hom_A(f, \mathcal{B}) \circ Hom_A(g, \mathcal{B})(\varphi)](x) \\ &= [Hom_A(f, \mathcal{B}) \circ Hom_A(g, \mathcal{B})](\varphi)(x). \end{aligned}$$

So we have $Hom_A(g \circ f, \mathcal{B}) = Hom_A(f, \mathcal{B}) \circ Hom_A(g, \mathcal{B})$.

* Let $\varphi: M \rightarrow \mathcal{B}$ a morphism of left B -algebras, $x \in M$, we have:

$$\begin{aligned} [Hom_A(1_M, \mathcal{B})(\varphi)](x) &= \varphi \circ 1_M(x) = \varphi(x) \Rightarrow \\ Hom_A(1_M, \mathcal{B})(\varphi) &= \varphi. \end{aligned}$$

So $Hom_A(1_M, \mathcal{B}) = 1_{Hom_A(M, \mathcal{B})}$ and therefore $Hom_A(-, \mathcal{B})$ is a contravariant functor. ■

Theorem 3.5 Let \mathcal{A} a A -bialgebra and S a central

saturated multiplicatively closed subset of A . Then it exists an isomorphism of left $S^{-1}A$ -algebras,

$$\mathcal{A} \otimes_A S^{-1}A \cong S^{-1}\mathcal{A}.$$

Proof

Consider $\psi_{\mathcal{A}}: \mathcal{A} \times S^{-1}A \rightarrow S^{-1}\mathcal{A}$

$$(x, \frac{a}{s}) \mapsto \frac{xa}{s}.$$

We have $\psi_{\mathcal{A}}$ is A -bilinear, so according to the universal property of the tensor product it exists a map A -linear $\delta_{\mathcal{A}}: \mathcal{A} \otimes S^{-1}A \rightarrow S^{-1}\mathcal{A}$ such that

$$\delta_{\mathcal{A}}(\sum x_i \otimes \frac{a_i}{s_i}) = \sum \frac{x_i a_i}{s_i}.$$

Let $\varphi_{\mathcal{A}}: S^{-1}\mathcal{A} \rightarrow \mathcal{A} \otimes S^{-1}A$

$$\frac{x}{s} \mapsto x \otimes \frac{1}{s}.$$

It is clear that $\varphi_{\mathcal{A}}$ is well defined.

We have

$$\begin{aligned} \varphi_{\mathcal{A}} \circ \delta_{\mathcal{A}}(\sum x_i \otimes \frac{a_i}{s_i}) &= \varphi_{\mathcal{A}}(\sum \frac{x_i a_i}{s_i}) = \sum \varphi_{\mathcal{A}}(\frac{x_i a_i}{s_i}) = \\ &= \sum x_i a_i \otimes \frac{1}{s_i} = \sum x_i \otimes \frac{a_i}{s_i}. \end{aligned}$$

So $\varphi_{\mathcal{A}} \circ \delta_{\mathcal{A}} = 1_{\mathcal{A} \otimes S^{-1}A}$, $\varphi_{\mathcal{A}}$ and $\delta_{\mathcal{A}}$ are inverse of each other.

Therefore $\mathcal{A} \otimes_A S^{-1}A \cong S^{-1}\mathcal{A}$. ■

Theorem 3.6 Let B a duo-ring, A a subring of B , S a central saturated multiplicatively closed subset of A , S_R the set of regular elements of S , \mathcal{A} a finitely presented right A -algebra and \mathcal{B} a $(B - A)$ -bialgebra. Then there exists the isomorphisms of left $S_R^{-1}B$ -algebras,

$$\begin{aligned} Hom_A(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B &\cong S_R^{-1}Hom_A(\mathcal{A}, \mathcal{B}) \\ &\cong Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}). \end{aligned}$$

Proof

• Since \mathcal{A} is a right A -algebra and \mathcal{B} is a $(B - A)$ -bialgebra, then after the proposition 3.1, $Hom_A(\mathcal{A}, \mathcal{B})$ is a left B -algebra, we have also $S_R^{-1}B$ is a $(S_R^{-1}B - B)$ -bialgebra, so after the proposition 3.2 and the theorem 3.5 we have,

$$Hom_A(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Hom_A(\mathcal{A}, \mathcal{B}).$$

• Show that $S_R^{-1}Hom_A(\mathcal{A}, \mathcal{B}) \cong Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is an isomorphism of left $S_R^{-1}B$ -algebras.

After [6] $S_R^{-1}Hom_A(\mathcal{A}, \mathcal{B}) \cong Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is an isomorphism of left $S_R^{-1}B$ -module and since $Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is a left $S_R^{-1}B$ -algebra, so by structure transport $S_R^{-1}Hom_A(\mathcal{A}, \mathcal{B})$ is a left $S_R^{-1}B$ -algebra.

Therefore $S_R^{-1}Hom_A(\mathcal{A}, \mathcal{B}) \rightarrow Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is the isomorphisms of left $S_R^{-1}B$ -algebras. ■

Theorem 3.7 Let B a ring, S a saturated multiplicatively closed subset of $Z(B)$, \mathcal{A} a finitely presented right $Z(B)$ -algebra and \mathcal{B} a $(B - Z(B))$ -bialgebra. Then there exists the isomorphisms of left $S^{-1}B$ -algebras,

$$\begin{aligned} \text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B}) \otimes_B S^{-1}B &\cong S^{-1}\text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B}) \\ &\cong \text{Hom}_{S^{-1}Z(B)}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}). \end{aligned}$$

Proof

• The first isomorphism is shown in the same way as that of the previous theorem.

• We have \mathcal{A} is a right $Z(B)$ -algebra $\Rightarrow S^{-1}\mathcal{A}$ is a right $S^{-1}Z(B)$ -algebra.

Also \mathcal{B} is a $(B - Z(B))$ -bialgebra $\Rightarrow S^{-1}\mathcal{B}$ a $(S^{-1}B - S^{-1}Z(B))$ -bialgebra. So $\text{Hom}_{S^{-1}Z(B)}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is a left $S^{-1}B$ -algebra and since there is a bijection between $S^{-1}\text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B})$ and $\text{Hom}_{S^{-1}Z(B)}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$, then by structure transport $S^{-1}\text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B}) \cong \text{Hom}_{S^{-1}Z(B)}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$ is an isomorphism of left $S^{-1}B$ -algebras. ■

Proposition 3.4 Let B a duo-ring, \mathcal{P} a prime ideal of B , \mathcal{A} a finitely presented right B -algebra, \mathcal{B} a $(B - B)$ -bialgebra, $S = (B - \mathcal{P}) \cap Z(B)$ and S_R the set of regular elements of S . Then it exists the isomorphisms of left $S^{-1}B$ -algebras,

$$\begin{aligned} \text{Hom}_B(\mathcal{A}, \mathcal{B}) \otimes_B S^{-1}B &\cong S^{-1}\text{Hom}_B(\mathcal{A}, \mathcal{B}) \\ &\cong \text{Hom}_{S^{-1}B}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}). \end{aligned}$$

Furthermore it exists the isomorphisms of left $S_R^{-1}B$ -algebras,

$$\begin{aligned} \text{Hom}_B(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B &\cong S_R^{-1}\text{Hom}_B(\mathcal{A}, \mathcal{B}) \\ &\cong \text{Hom}_{S^{-1}B}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}). \end{aligned}$$

Proof

Since B is a duo-ring, then $B - \mathcal{P}$ is saturated multiplicatively closed subset, so $S = (B - \mathcal{P}) \cap Z(B)$ is a central saturated multiplicatively closed subset of B .

So after the theorem 3.6 we have,

$$\begin{aligned} \text{Hom}_B(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B &\cong S_R^{-1}\text{Hom}_B(\mathcal{A}, \mathcal{B}) \cong \\ &\text{Hom}_{S^{-1}B}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}) \text{ and} \end{aligned}$$

$$\begin{aligned} \text{Hom}_B(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B &\cong S_R^{-1}\text{Hom}_B(\mathcal{A}, \mathcal{B}) \cong \\ &\text{Hom}_{S^{-1}B}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}). \quad \blacksquare \end{aligned}$$

Proposition 3.5 Let B a ring, \mathcal{P} a prime ideal of $Z(B)$, \mathcal{A} a finitely presented right B -algebra and \mathcal{B} a $(B - B)$ -bialgebra. Then it exists the isomorphisms of left $Z(B)_{\mathcal{P}}$ -algebras,

$$\begin{aligned} \text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B}) \otimes_B B_{\mathcal{P}} &\cong \text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B})_{\mathcal{P}} \\ &\cong \text{Hom}_{Z(B)_{\mathcal{P}}}(\mathcal{A}_{\mathcal{P}}, \mathcal{B}_{\mathcal{P}}). \end{aligned}$$

Proof

Since \mathcal{A} is a right B -algebra and \mathcal{B} a $(B - B)$ -bialgebra, then in particular \mathcal{A} is a right $Z(B)$ -algebra and \mathcal{B} a $(B - Z(B))$ -bialgebra.

Pose $S = Z(B) - \mathcal{P} \Rightarrow S$ is a saturated multiplicatively closed subset of $Z(B)$. So after the theorem 3.7 we have:

$$\begin{aligned} \text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B})_{\mathcal{P}} &= S^{-1}\text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B}) \cong \\ \text{Hom}_{S^{-1}Z(B)}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}) &= \text{Hom}_{Z(B)_{\mathcal{P}}}(\mathcal{A}_{\mathcal{P}}, \mathcal{B}_{\mathcal{P}}) \text{ and} \end{aligned}$$

$$\begin{aligned} \text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B}) \otimes_B B_{\mathcal{P}} &= \text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B}) \otimes_B S^{-1}B \cong \\ S^{-1}\text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B}) &= \text{Hom}_{Z(B)}(\mathcal{A}, \mathcal{B})_{\mathcal{P}}. \quad \blacksquare \end{aligned}$$

4. Functor $S^{-1}()$ and Functor $\text{Ext}_A(-, \mathcal{B})$ in the Category $A - \text{Alg}$ (resp. $\text{Alg} - A$)

4.1. Construction of the Derived Functor $\text{Ext}_A(-, \mathcal{B})$ in the Category $A - \text{Alg}$ (resp. $\text{Alg} - A$)

Theorem 4.1 Let \mathcal{A} a right A -algebra, \mathcal{B} a $(B - A)$ -bialgebra and

$P: P_n \xrightarrow{d_n} P_{n-1} \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathcal{A} \rightarrow 0$ a complex projective resolution of a right A -module \mathcal{A} . Then

$$[\text{Ker}d_{n+1}^*]/\langle \text{Im}d_n^* \rangle$$

is a unitary left B -algebra, where

$$d_n^* = \text{Hom}_A(d_n, \mathcal{B}): \text{Hom}_A(P_{n-1}, \mathcal{B}) \rightarrow \text{Hom}_A(P_n, \mathcal{B})$$

$$\varphi_n \mapsto \varphi_n \circ d_n$$

$[\text{Ker}d_{n+1}^*]$ is the unitary algebra generated by $\text{Ker}d_{n+1}^*$ and $\langle \text{Im}d_n^* \rangle$ is the ideal of $[\text{Ker}d_{n+1}^*]$ generated by $\text{Im}d_n^*$.

Proof

By applying the contravariant functor we have,

$$\begin{aligned} \text{Hom}_A(P, \mathcal{B}): 0 \rightarrow \text{Hom}_A(\mathcal{A}, \mathcal{B}) \rightarrow \\ \cdots \text{Hom}_A(P_{n-1}, \mathcal{B}) \xrightarrow{d_n^*} \text{Hom}_A(P_n, \mathcal{B}) \xrightarrow{d_{n+1}^*} \text{Hom}_A(P_{n+1}, \mathcal{B}). \end{aligned}$$

So after the proposition 3.1 $\text{Hom}_A(P_n, \mathcal{B})$ has a structure of left B -algebra, for all $n \geq 0$.

Show that d_n^* , for all $n \geq 0$, is a morphism of left B -algebras.

Let $\varphi, \psi \in \text{Hom}_A(P_{n-1}, \mathcal{B})$, $x \in P_n$.
whereas

$$\begin{aligned} [d_n^*(\varphi\psi)](x) &= [(\varphi\psi) \circ d_n](x) = \varphi(d_n(x))\psi(d_n(x)) \\ &= (\varphi \circ d_n)(x)(\psi \circ d_n)(x) \\ &= [(\varphi \circ d_n)(\psi \circ d_n)](x) \\ &= [d_n^*(\varphi)d_n^*(\psi)](x). \end{aligned}$$

So $d_n^*(\varphi\psi) = d_n^*(\varphi)d_n^*(\psi)$.

Let $\varphi \in \text{Hom}_A(P_{n-1}, \mathcal{B})$, $b \in B$, $x \in P_n$.
whereas

$$[d_n^*(b\varphi)](x) = [(b\varphi) \circ d_n](x) = (b\varphi)(d_n(x)) = b(\varphi \circ d_n)(x) = [b(d_n^*(\varphi))](x).$$

So $d_n^*(b\varphi) = b(d_n^*(\varphi))$.

Therefore d_n^* , for all $n \geq 0$, is a morphism of left B -algebras.

It should be noted that $Kerd_{n+1}^*$ is not in general a unitary algebra, so take $[Kerd_{n+1}^*]$ the subalgebra of the unitary left B -algebra $Hom_A(P_n, \mathcal{B})$ generated by $Kerd_{n+1}^*$.

On the other hand we have $Imd_n^* \subset Kerd_{n+1}^* \subset [Kerd_{n+1}^*]$.

It should be noted that Imd_n^* is not in general an ideal, so take the ideal $\langle Imd_n^* \rangle$ of $[Kerd_{n+1}^*]$ generated by Imd_n^* .

Therefore $[Kerd_{n+1}^*]/\langle Imd_n^* \rangle$ is a unitary left B -algebra. ■

In this paper we note by $Ext_A^n(\mathcal{A}, \mathcal{B}) = [Kerd_{n+1}^*]/\langle Imd_n^* \rangle$.

Proposition 4.1 *Let \mathcal{B} a $(B - A)$ -bialgebra. Then the correspondence*

$$Ext_A^n(-, \mathcal{B}): Mod - A \rightarrow B - Alg$$

(1). who has any right A -module M , we associate the left B -algebra $Ext_A^n(M, \mathcal{B})$,

(2). who has any morphism of right A -modules $f: M \rightarrow M'$, we associate $Ext_A^n(f, \mathcal{B}): Ext_A^n(M', \mathcal{B}) \rightarrow Ext_A^n(M, \mathcal{B})$ is a contravariant functor.

Proof

* After the theorem 4.1 we have $M \in Ob(Mod - A) \Rightarrow Ext_A^n(M, \mathcal{B}) \in Ob(B - Alg)$, so the action of $Ext_A^n(-, \mathcal{B})$ on the objects of $B - Alg$ makes sense.

* Let $f: M \rightarrow M'$ a morphism of right A -module. After the comparison theorem the following commutative diagram is obtained

$$\begin{array}{ccccccccccc} P_M : \dots & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \cdots & \longrightarrow & P_0 & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\ \bar{f} \downarrow & & \bar{f}_n \downarrow & & \bar{f}_{n-1} \downarrow & & & \bar{f}_0 \downarrow & & f \downarrow & & \\ P_{M'} : \dots & \longrightarrow & P'_n & \longrightarrow & P'_{n-1} & \cdots & \longrightarrow & P'_0 & \longrightarrow & \mathcal{B} & \longrightarrow & 0 \end{array}$$

By applying the contravariant functor $Hom_A(-, \mathcal{B})$ we have

$$\begin{array}{ccccccc} Hom_A(P_M, \mathcal{B}) : 0 & \longrightarrow & Hom_A(M, \mathcal{B}) & \longrightarrow & Hom_A(P_0, \mathcal{B}) & \cdots & \\ Hom_A(\bar{f}, \mathcal{B}) \downarrow & & Hom_A(f, \mathcal{B}) \downarrow & & Hom_A(\bar{f}_0, \mathcal{B}) \downarrow & & \\ Hom_A(P_{M'}, \mathcal{B}) : 0 & \longrightarrow & Hom_A(M', \mathcal{B}) & \longrightarrow & Hom_A(P'_0, \mathcal{B}) & \cdots & \end{array}$$

So $Hom_A(\bar{f}, \mathcal{B}): Hom_A(P_M, \mathcal{B}) \rightarrow Hom_A(P_{M'}, \mathcal{B})$ is a morphism of chain complex.

Whereas

$$H_n(Hom_A(\bar{f}, \mathcal{B})): H_n(Hom_A(P_M, \mathcal{B})) \rightarrow H_n(Hom_A(P_{M'}, \mathcal{B}))$$

$$\bar{z}_n \mapsto \overline{Hom_A(\bar{f}_n, \mathcal{B})z_n}.$$

After the theorem 4.1 $Hom_A(f, \mathcal{B}) = f^*$ and $Hom_A(\bar{f}_n, \mathcal{B}) = \bar{f}_n^*$ are morphism of left B -algebras.

So $H_n(Hom_A(\bar{f}, \mathcal{B})) = Ext_A^n(f, \mathcal{B})$ is a morphism of left B -algebra, so the action of $Ext_A^n(-, \mathcal{B})$ on the arrow makes sense.

* Whereas

$$Ext_A^n(g \circ f, \mathcal{B}) = H_n(Hom_A(\bar{g} \circ f, \mathcal{B})) = H_n(Hom_A(\bar{g} \circ \bar{f}, \mathcal{B}))$$

$$= H_n[Hom_A(\bar{f}, \mathcal{B}) \circ Hom_A(\bar{g}, \mathcal{B})]$$

$$= H_n(Hom_A(\bar{f}, \mathcal{B})) \circ H_n(Hom_A(\bar{g}, \mathcal{B}))$$

$$= Ext_A^n(f, \mathcal{B}) \circ Ext_A^n(g, \mathcal{B}).$$

* Whereas

$$Ext_A^n(1_M, \mathcal{B})(\bar{z}_n) = \overline{Hom_A((\bar{1}_M)_n, \mathcal{B})z_n} = \overline{1_{Hom_A(M, \mathcal{B})}z_n} = \bar{z}_n.$$

So $Ext_A^n(1_M, \mathcal{B}) = 1_{Ext_A^n(M, \mathcal{B})}$.

Therefore $Ext_A^n(-, \mathcal{B}): Mod - A \rightarrow B - Alg$ is a contravariant functor. ■

Theorem 4.2 *Let B a duo-ring, A a subring of B , S a central saturated multiplicatively closed subset of A , S_R the set of regular elements of S and \mathcal{B} a $(B - A)$ -bialgebra. Then the correspondence*

$$Ext_{S^{-1}A}^n(-, S^{-1}\mathcal{B}): Mod - S^{-1}A \rightarrow S_R^{-1}B - Alg$$

is a contravariant functor.

Proof

just see that after the theorem 3.2, $S^{-1}\mathcal{B}$ is a $(S_R^{-1}B - S^{-1}A)$ -bialgebra. ■

Corollary 4.1 *Let B a duo-ring, A a subring of B , S a central saturated multiplicatively closed subset of A , S_R the set of regular elements of S and \mathcal{B} a $(B - A)$ -bialgebra. Then the correspondence*

$$Ext_{S^{-1}A}^n(-, S^{-1}\mathcal{B}): Alg - S^{-1}A \rightarrow S_R^{-1}B - Alg$$

is a contravariant functor.

Proof

Whereas $S^{-1}\mathcal{A} \in Ob(Alg - S^{-1}A) \Rightarrow S^{-1}\mathcal{A} \in Ob(Mod - S^{-1}A)$, so the conditions of the theorem 4.2 are verified. ■

4.2. Relationship Between Functor $S^{-1}()$ and Functor $Ext_A(-, \mathcal{B})$ in Category $A - Alg$ (resp. $Alg - A$)

Theorem 4.3 *Let B a noetherian duo-ring, A a subring of B , S a central saturated multiplicatively closed subset of A , S_R the set of regular elements of S , \mathcal{A} a finitely generated right A -algebra and \mathcal{B} a $(B - A)$ -bialgebra. Then it exists the isomorphisms of left $S_R^{-1}B$ -algebras,*

$$Ext_A^n(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Ext_A^n(\mathcal{A}, \mathcal{B}) \cong Ext_{S^{-1}A}^n(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

Proof

After the theorem 3.5 we have the following isomorphism of left $S_R^{-1}B$ -algebra $Ext_A^n(\mathcal{A}, \mathcal{B}) \otimes_B S_R^{-1}B \cong S_R^{-1}Ext_A^n(\mathcal{A}, \mathcal{B})$.

* Since B is a noetherian ring and \mathcal{A} is a finitely generated left B -algebra, then \mathcal{A} admits a projective resolution P .

Also since \mathcal{A} is a finitely generated left algebra over a noetherian ring B , then \mathcal{A} is finitely presented.

So after the theorem 3.6 we have the following isomorphism of left $S_R^{-1}B$ -algebra,

$$S_R^{-1}Hom_A(\mathcal{A}, \mathcal{B}) \cong Hom_{S^{-1}A}(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

So we can deduce the following complex isomorphism:

$$S_R^{-1}Hom_A(P_{\mathcal{A}}, \mathcal{B}) \cong Hom_{S^{-1}A}(S^{-1}(P_{\mathcal{A}}), S^{-1}\mathcal{B}).$$

So,

$$H_n(S_R^{-1}Hom_A(P_{\mathcal{A}}, \mathcal{B})) \cong H_n(Hom_{S^{-1}A}(S^{-1}(P_{\mathcal{A}}), S^{-1}\mathcal{B})).$$

Since the homological functor H_n commutes with the functor $S^{-1}()$, then we have on the one hand,

$$H_n(S_R^{-1}Hom_A(P_{\mathcal{A}}, \mathcal{B})) \cong S_R^{-1}H_n(Hom_A(P_{\mathcal{A}}, \mathcal{B})) = S_R^{-1}Ext_A^n(\mathcal{A}, \mathcal{B}).$$

On the other hand we have,

$$H_n(Hom_{S^{-1}A}(S^{-1}(P_{\mathcal{A}}), S^{-1}\mathcal{B})) = Ext_{S^{-1}A}^n(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

Therefore $S_R^{-1}Ext_A^n(\mathcal{A}, \mathcal{B}) \cong Ext_{S^{-1}A}^n(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$. ■

Corollary 4.2 Let B a noetherian duo-ring, \mathcal{P} a prime ideal of B , \mathcal{A} a finitely generated right B -algebra, \mathcal{B} a $(B - \mathcal{P})$ -bialgebra, $S = (B - \mathcal{P}) \cap Z(B)$ and S_R the set of regular elements of $B - \mathcal{P}$. Then it exists the isomorphisms of left $S^{-1}B$ -algebras,

$$S_R^{-1}Ext_B^n(\mathcal{A}, \mathcal{B}) \cong Ext_{S^{-1}B}^n(S^{-1}\mathcal{A}, S^{-1}\mathcal{B}).$$

Proof

Since B is a duo-ring, then $B - \mathcal{P}$ is saturated multiplicatively closed subset, so $S = (B - \mathcal{P}) \cap Z(B)$ is a central saturated multiplicatively closed subset of B .

So after the theorem 4.3 we have, $S_R^{-1}Ext_B^n(\mathcal{A}, \mathcal{B}) \cong Ext_{S^{-1}B}^n(S^{-1}\mathcal{A}, S^{-1}\mathcal{B})$. ■

5. Conclusion

In this work the functors $Hom_A(-, \mathcal{B}): Alg - A \rightarrow B - Alg$, $- \otimes_A \mathcal{B}: B - Alg \rightarrow Alg - A$ and $Ext_A^n(-, \mathcal{B}): Alg - A \rightarrow B - Alg$ are built and their relationship with the functor $S^{-1}()$ has been studied. The following algebra structures: $Hom_A(M, \mathcal{B})$, $Ext_A^n(M, \mathcal{B})$, $Hom_A(\mathcal{A}, \mathcal{B})$, $\mathcal{A} \otimes_A \mathcal{B}$ and $Ext_A^n(\mathcal{A}, \mathcal{B})$ has been defined where M is a module and \mathcal{A} and \mathcal{B} are algebras.

The covariant functors $Hom_A(\mathcal{A}, -)$, $\mathcal{A} \otimes_A -$ and $Ext_A^n(\mathcal{A}, -)$ will be studied in the category $A - Alg$ (resp. $Alg - A$) and their relationship with the functor $S^{-1}()$ will be studied. In the future the result in [6] on the adjunction between the functors $Ext_A^n(\mathcal{A}, -)$ and $Tor_n^A(\mathcal{A}, -)$ (resp.

$Ext_A^n(-, \mathcal{B})$ and $Tor_n^A(-, \mathcal{B})$) done in the category of $A - Mod$ will be generalized in the category $A - Alg$.

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